ADDENDA:
COMPUTING EUCLIDEAN BELYI MAPS

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This note gives an addenda for the article *Computing Euclidean Belyi maps* [1].

1. Addenda

The addenda is summarized in the following additional remark.

Remark 3.2.10. *If in* Algorithm 2.4.4 *we compute instead the* Smith *normal form (SNF) of A as* $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} = PAQ$ *(with $n \mid m$), the result gives a basis for Λ_Γ relative to a basis for Λ_Δ such that $\Lambda_\Gamma = \langle n\omega_1', m\omega_2' \rangle$ with $\Lambda_\Delta = \langle \omega_1', \omega_2' \rangle$. Accordingly, we adjust Step 4 in* Algorithm 3.2.5 *by replacing the occurrences of ω_1 and ω_2 respectively with $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ where* $Q^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Incorporating Remark 3.2.9*, we may further simplify by factoring n from each entry in our basis matrix (corresponding to factoring the multiplication by n map from $\widehat{\psi}$). This reduces us to the case $n = 1$ in* Algorithm 3.2.5.

In more detail, computing the map $\psi\colon E(\Gamma) \to E(\Delta)$ is the most complicated and costly step in Algorithm 3.5.1. To do so, we must first determine a basis for the lattice Λ_Γ relative to a basis for the lattice Λ_Δ. In Corollary 2.2.7, we make a "standard" choice for the basis vectors ω_1 and ω_2 for Λ_Δ that coincide with the periods Magma assigns to our canonical curves $E_{\hexagon}$ and $E_{\square}$. Algorithm 2.4.4 then produces a two column matrix A whose rows, taken as coordinates relative to the basis vectors ω_1 and ω_2, give a set of vectors that span Λ_Γ.

Reducing A to Hermite normal form and taking its first two rows gives a basis matrix

$$B_H := \begin{pmatrix} n_1 & n_2 \\ 0 & m_2 \end{pmatrix}$$

such that $\Lambda_\Gamma = \langle n_1\omega_1 + n_2\omega_2, m_2\omega_2 \rangle$.

If, instead, we reduce A to *Smith* normal form and take its first two rows, we obtain a matrix of the form

$$B_S := \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$$

where n divides m. Like with B_H, the matrix B_S describes a basis for Λ_Γ relative to a basis for Λ_Δ such that $\Lambda_\Gamma = \langle n\omega_1', m\omega_2' \rangle$ with $\Lambda_\Delta = \langle \omega_1', \omega_2' \rangle$. We note that the basis vectors ω_1' and ω_2' need not be the same as ω_1 and ω_2.

Because Magma's implementation of the Weierstrass $\wp$-function takes inputs relative to ω_1 and ω_2, it is then necessary to relate ω_1' and ω_2' back to the "standard" basis vectors. Let $P, Q \in \mathrm{GL}_2(\mathbb{Z})$ be such that $B_S = PB_HQ$. Then the matrices P and Q correspond, respectively, to elementary row and column operations performed on B_H that transform it to B_S. As each elementary column operation corresponds to an invertible change to the choice of basis for Λ_Δ, we can recover the relationship between each ω_i' and ω_i from the matrix Q indicated above. Specifically, if

$$Q^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$.

Working with B_S rather than B_H simplifies our computation of the isogeny $\psi : E(\Gamma) \to E(\Delta)$. Algorithm 3.2.5 describes a procedure for computing ψ (by first computing its dual, $\widehat{\psi}$) that assumes the Hermite basis matrix B_H. If we instead work with the Smith matrix B_S, we may assume that $n_2 = 0$ and let $n_1 = n$ and $m_2 = m$. Incorporating remark 3.2.9 and recalling that n divides m, we may further simplify by factoring n from each entry in our basis matrix (corresponding to factoring the multiplication by n map from $\widehat{\psi}$), leaving us with the matrix

$$\frac{1}{n}B_S = \begin{pmatrix} 1 & 0 \\ 0 & m/n \end{pmatrix}$$

where $m/n \in \mathbb{Z}$.

The combined effect of Remark 3.2.9 and this Smith simplification allows us to always assume in Algorithm 3.2.5 a basis matrix B of particularly simple form:

$$B := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$$

This basis matrix gives coordinates relative to ω_1' and ω_2' rather than ω_1 and ω_2. Accordingly, we adjust the implementation of step 4 in Algorithm 3.2.5 by replacing the occurrences of ω_1 and ω_2 respectively with $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ as obtained above.

References

[1] Matt Radosevich and John Voight, *Computing Euclidean Belyi maps*, accepted to J. Théorie Nombres Bordeaux.

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