# ERRATA AND ADDENDA: *COMPUTING EUCLIDEAN BELYI MAPS*

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This note gives errata and addenda for the article *Computing Euclidean Belyi maps* [1].

## 1. Errata

(1) Remark 4.2.4: the map should be
$$\varphi(x) = 36(\zeta_6 - 1)\frac{(x-2)(x-2\zeta_6-1)^2(x^2+2x-11)}{(x+2\zeta_6-3)^6}$$
(so $z = \zeta = \zeta_6$).

## 2. Addenda

The addenda is summarized in the following additional remark.

**Remark 3.2.10.** *If in* Algorithm 2.4.4 *we compute instead the* Smith *normal form (SNF) of $A$ as $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} = PAQ$ (with $n \mid m$), the result gives a basis for $\Lambda_\Gamma$ relative to a basis for $\Lambda_\Delta$ such that $\Lambda_\Gamma = \langle n\omega_1', m\omega_2'\rangle$ with $\Lambda_\Delta = \langle \omega_1', \omega_2'\rangle$. Accordingly, we adjust Step 4 in* Algorithm 3.2.5 *by replacing the occurrences of $\omega_1$ and $\omega_2$ respectively with $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ where $Q^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.*

*Incorporating* Remark 3.2.9*, we may further simplify by factoring $n$ from each entry in our basis matrix (corresponding to factoring the multiplication by $n$ map from $\widehat{\psi}$). This reduces us to the case $n = 1$ in* Algorithm 3.2.5.

In more detail, computing the map $\psi\colon E(\Gamma) \to E(\Delta)$ is the most complicated and costly step in Algorithm 3.5.1. To do so, we must first determine a basis for the lattice $\Lambda_\Gamma$ relative to a basis for the lattice $\Lambda_\Delta$. In Corollary 2.2.7, we make a "standard" choice for the basis vectors $\omega_1$ and $\omega_2$ for $\Lambda_\Delta$ that coincide with the periods Magma assigns to our canonical curves $E_{\hexagon}$ and $E_{\square}$. Algorithm 2.4.4 then produces a two column matrix $A$ whose rows, taken as coordinates relative to the basis vectors $\omega_1$ and $\omega_2$, give a set of vectors that span $\Lambda_\Gamma$.

Reducing $A$ to Hermite normal form and taking its first two rows gives a basis matrix
$$B_H := \begin{pmatrix} n_1 & n_2 \\ 0 & m_2 \end{pmatrix}$$
such that $\Lambda_\Gamma = \langle n_1\omega_1 + n_2\omega_2, m_2\omega_2\rangle$.

If, instead, we reduce $A$ to *Smith* normal form and take its first two rows, we obtain a matrix of the form
$$B_S := \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$$

where $n$ divides $m$. Like with $B_H$, the matrix $B_S$ describes a basis for $\Lambda_\Gamma$ relative to a basis for $\Lambda_\Delta$ such that $\Lambda_\Gamma = \langle n\omega_1', m\omega_2'\rangle$ with $\Lambda_\Delta = \langle \omega_1', \omega_2'\rangle$. We note that the basis vectors $\omega_1'$ and $\omega_2'$ need not be the same as $\omega_1$ and $\omega_2$.

Because Magma's implementation of the Weierstrass $\wp$-function takes inputs relative to $\omega_1$ and $\omega_2$, it is then necessary to relate $\omega_1'$ and $\omega_2'$ back to the "standard" basis vectors. Let $P, Q \in \mathrm{GL}_2(\mathbb{Z})$ be such that $B_S = PB_HQ$. Then the matrices $P$ and $Q$ correspond, respectively, to elementary row and column operations performed on $B_H$ that transform it to $B_S$. As each elementary column operation corresponds to an invertible change to the choice of basis for $\Lambda_\Delta$, we can recover the relationship between each $\omega_i'$ and $\omega_i$ from the matrix $Q$ indicated above. Specifically, if

$$Q^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$.

Working with $B_S$ rather than $B_H$ simplifies our computation of the isogeny $\psi : E(\Gamma) \to E(\Delta)$. Algorithm 3.2.5 describes a procedure for computing $\psi$ (by first computing its dual, $\widehat{\psi}$) that assumes the Hermite basis matrix $B_H$. If we instead work with the Smith matrix $B_S$, we may assume that $n_2 = 0$ and let $n_1 = n$ and $m_2 = m$. Incorporating remark 3.2.9 and recalling that $n$ divides $m$, we may further simplify by factoring $n$ from each entry in our basis matrix (corresponding to factoring the multiplication by $n$ map from $\widehat{\psi}$), leaving us with the matrix

$$\frac{1}{n} B_S = \begin{pmatrix} 1 & 0 \\ 0 & m/n \end{pmatrix}$$

where $m/n \in \mathbb{Z}$.

The combined effect of Remark 3.2.9 and this Smith simplification allows us to always assume in Algorithm 3.2.5 a basis matrix $B$ of particularly simple form:

$$B := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$$

This basis matrix gives coordinates relative to $\omega_1'$ and $\omega_2'$ rather than $\omega_1$ and $\omega_2$. Accordingly, we adjust the implementation of step 4 in Algorithm 3.2.5 by replacing the occurrences of $\omega_1$ and $\omega_2$ respectively with $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ as obtained above.

## References

[1] Matt Radosevich and John Voight, *Computing Euclidean Belyi maps*, accepted to J. Théorie Nombres Bordeaux.

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