ERRATA:
COMPUTING FUNDAMENTAL DOMAINS
FOR FUCHSIAN GROUPS

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This note gives errata for the article *Computing fundamental domains for Fuchsian groups* [4]. The author would like to thank Eran Assaf, Gebhard Boeckle, Aurel Page, Jeroen Sijsling, and Charles Stibitz for finding these mistakes.

(1) Proposition 1.1 is incorrect as stated. The corrected statement is as follows.

Let $D \subset \mathfrak{H}$ be a hyperbolic polygon. For a vertex v of D, we denote by $\vartheta_D(v)$ the interior angle of D at v. Let P be a side pairing for D. We say that P *satisfies the cycle condition* if for every cycle C of vertices in D under P there exists $e \in \mathbb{Z}_{>0}$ such that

$$\sum_{v \in C} \vartheta_D(v) = \frac{2\pi}{e}.$$

Proposition 1.1. *The Dirichlet domain $D(p)$ has a side pairing P, and the set $G(P)$ generates Γ. Conversely, let $D \subset \mathfrak{H}$ be a hyperbolic polygon and let P be a side pairing for D which satisfies the cycle condition. Then D is a fundamental domain for the group generated by $G(P)$.*

Proof. One must verify Beardon's condition (A6) [1, p. 246] or (A6)' [1, p. 249], which formalizes the equivalent angle condition (g) given by Maskit [3, p. 223]. □

This mistake does not affect any other result in the paper; it does however affect the proof of correctness of Algorithm 4.7. The following should be added after the second paragraph of the proof.

Proof of correctness, Algorithm 4.7. We must argue that the output $D = \operatorname{ext}(U)$ of Algorithm 4.7 satisfies the cycle condition. Let C be a cycle of vertices in D. Consider small neighborhoods of each vertex in C in D. If these neighborhoods are disjoint under the action of Γ, then they glue to give a neighborhood in the quotient $\Gamma\backslash\mathfrak{H}$, hence the cycle condition holds for C. Making these neighborhoods smaller, we may assume that each $v \in C$ is an elliptic fixed point. But then by Proposition 5.4 (which applies equally well to exterior domains) and the accompanying discussion, we may assume that the elliptic cycle has length 1, and consequently the cycle condition is trivially satisfied. □

An alternative method would be to ensure that the cycle condition holds, and modify Algorithm 4.7 as follows.

Date: April 1, 2021.

5. If all vertices of $E = \operatorname{ext}(U)$ are paired, proceed to Step 6. Otherwise, for each $g \in G$ with a vertex $v \in I(g)$ which is not paired, compute $\overline{g} := \operatorname{red}_G(g; v)$, where if v is a vertex at infinity we replace v by a nearby point in $I(g^{-1}) \setminus E \subset \mathfrak{D}$. Add the reductions $\overline{g}$ for each nonpaired vertex v to G and return to Step 2.
6. Run Algorithm 5.2, and let S be the set of minimal cycles $g \neq 1$ with a fixed point in $\mathfrak{D}$. If $S \subseteq U$, return U; otherwise, set $U := U \cup S \cup S^{-1}$ and return to Step 2.

Following as in the proof above, in the third paragraph we have checked explicitly that the elliptic fixed point $v \in C$ has a neighborhood of the right size, with edges contained in $I(g)$ and $I(g^{-1})$ where g is the minimal cycle fixing v.

Example. Consider the elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $W = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$. We have $W = STS$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and S, T generate $\mathrm{SL}_2(\mathbb{Z})$. Since $(ST)^3 = 1$ we conclude that $WTW = S$ so the group generated by W and T is $\mathrm{SL}_2(\mathbb{Z})$. The polygon D with vertices at $i\infty, \rho, 0, -\rho^2$ where $\rho = (-1+\sqrt{-3})/2$ is the union of the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ along with its translate by S. At the same time, P has a side pairing given by W and T.

The side pairing P does not satisfy the cycle condition, as ρ and $-\rho^2$ are identified but the sum of the interior angles is $2 \cdot 2 \cdot (2\pi/6) = 4\pi/3$. However, this discrepancy is caught in Step 3 of Algorithm 4.7: taking $p = 2i$ for example, we have $G = G' = \{W, T, W^{-1}, T^{-1}\}$ and $\operatorname{red}_{\{T, W^{-1}, T^{-1}\}}(W) = TW = (TS)^2$ since $\operatorname{Re}(Wp) = -4/5 < -1/2$. From this, the algorithm quickly recovers the side pairing elements T and S.

Example. On the other hand, if we start with the point $p = i/2$ and the generators S, T, then running through Algorithm 4.7 one makes it to Step 5 with exterior domain the region given by $|\operatorname{Re}(z)| \leq 1/2$ and $|z| \leq 1$ (not ≥ 1!); the vertices are paired, but this region does not have finite hyperbolic area.

Indeed, the internal angles at the vertices $\rho = (-1/2+\sqrt{-3})/2$ and $\rho+1$ are $2\pi/3$ and hence the cycle condition is not satisfied.

However, if in following the amended proof above, one changes the point p outside the exceptional set (described in Proposition 5.4), then one recovers the fundamental domain. For example, consider $p = \epsilon + i/2$ where $\epsilon > 0$. Then the fixed point $(1+\sqrt{-3})/2$ is now closer to p, the vertices are not paired, and running through Algorithm 4.7, we find a new side-pairing element. Letting $\epsilon \to 0$, one recovers the fundamental domain above. (The set of side-pairing elements is discrete, so if you perturb the center p continuously by a small amount, these elements do not change; so to recover the Dirichlet domain you can just take the exterior domain of the generators computed using the perturbed center p.)

This case is also caught by the added Step 6 in Algorithm 4.7: we compute that ST^{-1} fixes ρ but is not in the set $U = \{S, T\}$; adding ST^{-1} and TS^{-1} to U and continuing then yields the correct domain.

(2) Remark 1.5: the result as stated is for g^{-1}, not g: the correct perpendicular bisector is the half-circle of square radius $\dfrac{a^2+b^2+c^2+d^2-2}{(c^2+d^2-1)^2}$ centered at $\dfrac{ac+bd}{c^2+d^2-1} \in \mathbb{R}$.

(3) Section 2, "the inital and terminal points are normalized so that the path along L follows a counterclockwise orientation": this is to be interpreted as clockwise around the origin.

(4) Step 2 of Algorithm 2.5 needs to be changed to:

2. For each $g \in G$, compute

$$\theta_g := \begin{cases} \arg(I(g) \cap L), & \text{if } I(g) \cap L \neq \emptyset; \\ \arg(\operatorname{in}(I(g))), & \text{if } I(g) \cap L = \emptyset. \end{cases}$$

Let

$$\theta' := \min\{\theta_g : g \in G \text{ and } \theta_g \geq \theta\}$$

and $H := \{g \in G : \theta_g = \theta'\}$.

a. Suppose (every) $g \in H$ has $I(g) \cap L \neq \emptyset$. If $L = [0, 1]$, let $g \in H$ minimize $I(g) \cap L$. Otherwise, let $g \in H$ minimize $\angle(L, I(g))$.

b. Suppose (every) $g \in H$ has $I(g) \cap L = \emptyset$. Let $g \in H$ maximize the radius of $I(g)$.

Let $U := U \cup \{g\}$ and let $L := I(g) \cap \overline{\mathfrak{D}}$ and let $\theta := \theta'$.

This deals with the case when more than one g exists, in the first execution of step 2 (all intersections with L have $\theta = 0$), and to ensure that all candidate vertices are traversed in counterclockwise order. In the example in Figure 2.4, after the first iteration where $I(g_1)$ is found, to get $I(g_2)$ we need the next vertex to be v_2 (improper), not to take the first proper intersection.

(5) In Step 3 of Algorithm 4.7, it should read "$\overline{g} = \operatorname{red}_U(g)$", and then it suffices to loop over $g \in G$ such that $g^{-1} \notin U$. Indeed, Steps 3 and 4 use reduction to eliminate extraneous generators; the only essential fact used in the proof of correctness is that the group generated by the set G does not change. (So for that matter, one may in Step 4 always set $G := G'$ before returning to Step 3 to avoid duplicate computation.)

(6) In the proof of correctness for Algorithm 4.7, it should say "Proposition 1.1", not "Theorem 1.1".

(7) The definition of *accidental cycle* on page 484 is delayed and logically should occur before Proposition 5.4.

(8) Proof of Lemma 5.6: Each occurrence of H_0 should be H_2. The Mayer-Vietoris sequence [2, Corollary II.7.7] is

$$0 = H_2(A, \mathbb{Z}) \to H_2(\Gamma_1, \mathbb{Z}) \oplus H_2(\Gamma_2, \mathbb{Z}) \to H_2(\Gamma, \mathbb{Z}) \to H_1(A, \mathbb{Z}) = 0$$

with A the trivial group. We have $H_2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ and if Γ_i is nontrivial then $H_2(\Gamma_i, \mathbb{Z}) \cong \mathbb{Z}$ as well [2, §II.4], so at least one Γ_i is trivial, which is the result.

(9) In Step 1 of Algorithm 5.7, replace G by H: "Let $H \subset G(P)$ be such that $g \in H$ implies either $g = g^{-1}$ or $g^{-1} \notin H$."

(10) The term "finite coarea" is preferred to "cofinite area" and should be replaced everywhere.

References

[1] Alan F. Beardon, *The geometry of discrete groups*, Grad. Texts in Math., vol. 91, Springer-Verlag, New York, 1995.

[2] Kenneth S. Brown, *Cohomology of groups*, Grad. Texts in Math., vol. 87, Springer-Verlag, New York, 1982.

[3] Bernard Maskit, *On Poincaré's theorem for fundamental polygons*, Advances in Math. **7** (1971), 219–230.

[4] John Voight, *Computing fundamental domains for cofinite Fuchsian groups*, J. Théorie Nombres Bordeaux **21** (2009), no. 2, 467–489.