THE GAUSS HIGHER RELATIVE CLASS NUMBER PROBLEM

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ABSTRACT. Assuming the 2-adic Iwasawa main conjecture, we find all CM fields with higher relative class number at most 16: there are at least 31 and at most 34 such fields, and at most one is nonabelian.

The problem of determining all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ of small class number $h(K)$ was first posed in Article 303 of Gauss' *Disquisitiones Arithmeticae*. It would take almost 150 years of work, culminating in the results of Stark [19] and Baker [1], to determine those fields with class number at most two: there are exactly 27, the last having discriminant $d = -427$. (See Goldfeld [6] or Stark [21] for a history of this problem.) Significant further progress has been made recently by Watkins [24], who enumerated all such fields K with class number $h(K) \leq 100$.

One interesting generalization of the Gauss class number problem is to replace $\mathbb{Q}$ by a totally real field F. Let K/F be a CM extension, i.e., K is a totally imaginary quadratic extension of a totally real field F, and let $[F : \mathbb{Q}] = n$. We have the divisibility relation $h(F) \mid h(K)$, and we denote by $h^-(K) = h(K)/h(F)$ the relative class number. It is known that there are only finitely many CM fields K with degree $[K : \mathbb{Q}] = 2n$ and fixed relative class number [20]. The complete list of CM fields of relative class number one is still unknown (see Lee-Kwon [11] for an overview). However, many partial results are known: for example, there are exactly 302 imaginary abelian number fields K with relative class number one [3], each having degree $[K : \mathbb{Q}] \leq 24$.

The integer $h^-(K)$ can be determined by the analytic relative class number formula, as follows. Let $\chi : \mathrm{Gal}(K/F) \to \{\pm 1\}$ denote the nontrivial character associated to the extension K/F and let $L(\chi, s)$ denote the Artin L-function associated to χ. Then

$$L(\chi, 0) = \frac{2^n}{Q(K)} \frac{h^-(K)}{w(K)},$$

where $w(K) = \#\mu(K)$ is the number of roots of unity in the field K and $Q(K) = [\mathbb{Z}_K^* : \mathbb{Z}_F^* \, \mu(K)] \in \{1, 2\}$ is the Hasse Q-unit index.

In this article, we consider the further generalization of the Gauss problem to higher relative class numbers of CM fields. Let E be a number field with ring of integers $\mathbb{Z}_E$ and let $m \in \mathbb{Z}_{\geq 3}$ be an odd integer. We define the *higher class group* of E to be

$$H^2(\mathrm{Spec}\, \mathbb{Z}_E, \mathbb{Z}(m)) = \prod_p H^2_{\text{ét}}(\mathrm{Spec}\, \mathbb{Z}_E[1/p], \mathbb{Z}_p(m)).$$

The group $H^2(\mathrm{Spec}\, \mathbb{Z}_E, \mathbb{Z}(m))$ is finite, and we let $h_m(E)$ denote its order. For $p \neq 2$, the Quillen-Lichtenbaum conjecture (which appears to have been proven by Voevodsky-Rust-Suslin-Weibel) implies that the étale ℓ-adic Chern character

$$K_{2m-2}(\mathbb{Z}_E) \otimes \mathbb{Z}_p \xrightarrow{\sim} H^2_{\text{ét}}(\mathrm{Spec}\, \mathbb{Z}_E[1/p], \mathbb{Z}_p(m))$$

is an isomorphism, thus $h_m(E)$ and $\#K_{2m-2}$ agree up to a power of two. (See §1 for more detail.)

For the CM extension K/F, we define the *higher relative class number* to be

$$h_m^-(K) = \frac{h_m(K)}{h_m(F)} \in \mathbb{Z}.$$

In analogy with the usual class number, an analytic higher relative class number formula holds: up to a power of 2, we have the equality

$$L(\chi, 1-m) = (-1)^{n\left(\frac{m-1}{2}\right)} \frac{2^{n+1}}{Q_m(K)} \frac{h_m^-(K)}{w_m(K)}, \tag{1}$$

where $w_m(K) \in \mathbb{Z}_{>0}$ and $Q_m(K) \in \{1,2\}$ are the number of higher roots of unity and the higher Q-index, respectively. Assuming the Iwasawa main conjecture for $p = 2$, the formula (1) holds exactly (see Kolster [9] and the discussion below).

Our main result is as follows.

Theorem. *Suppose that the Iwasawa main conjecture holds for $p = 2$. Then there are at least 31 and at most 34 pairs $(K/F, m)$ where $h_m^-(K) \leq 16$.*

The extensions are listed in Tables 4.1–4.2 in Section 4, and explains the calculations of Henderson ($h_m^-(K) = 1$ and $F = \mathbb{Q}$) reported in Kolster [9, §3].

We begin in Section 1 by giving the necessary background. In Section 2, we estimate the size of the higher relative class number using the analytic formula in an elementary way, and prove a statement in the spirit of the Brauer-Siegel theorem. In Section 3, we combine these estimates with the Odlyzko bounds to reduce the problem to a finite computation and then carry it out to prove our main result. We conclude in Section 4 by tabulating the fields.

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1. Background

In this section, we state in detail the higher relative class number formula, which determines analytically the order of the higher relative class number of a CM extension of number fields.

We begin by introducing the L-function and its functional equation—see Lang [10, Chapter XIII] or Neukirch [15, Chapter VII] for a reference. Let E be a number field with ring of integers $\mathbb{Z}_E$, absolute discriminant d_E, and degree $[E : \mathbb{Q}] = r_1 + 2r_2 = n$, where r_1, r_2 denote the number of real and pairs of complex places of E. Let $\zeta_E(s)$ be the Dedekind zeta function of E, defined by

$$\zeta_E(s) = \sum_{\mathfrak{a} \subset \mathbb{Z}_E} \frac{1}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p} \subset \mathbb{Z}_E} \left(1 - \frac{1}{(N\mathfrak{p})^s}\right)^{-1}$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Then $\zeta_E(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. Define the completed zeta function of E by

$$\xi_E(s) = \left(\frac{d_E}{4^{r_2}\pi^n}\right)^{s/2} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_E(s).$$

Then ξ_E satisfies the functional equation $\xi_E(1-s) = \xi_E(s)$, and it follows that

$$\zeta_E(1-s) = \zeta_E(s) \left(\frac{d_E}{4^{r_2}\pi^n}\right)^{s-1/2} \frac{\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}}{\Gamma((1-s)/2)^{r_1}\Gamma(1-s)^{r_2}}. \tag{2}$$

Now let K/F be a CM extension of number fields with $[F:\mathbb{Q}]=n$ and let χ denote the nontrivial character of $\mathrm{Gal}(K/F)$. Then the Artin L-function

$$L(\chi,s)=\frac{\zeta_K(s)}{\zeta_F(s)} \tag{3}$$

has an analytic continuation to $\mathbb{C}$, and for $s\in\mathbb{C}$ with $\operatorname{Re} s>1$ we have

$$L(\chi,s)=\sum_{\mathfrak{a}\subset\mathbb{Z}_F}\frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s}=\prod_{\mathfrak{p}\subset\mathbb{Z}_F}\left(1-\frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1}. \tag{4}$$

Applying equation (2) to (3) we obtain after simplification that

$$L(\chi,1-s)=L(\chi,s)\left(\frac{1}{(4\pi)^n}\frac{d_K}{d_F}\right)^{s-1/2}\left(\frac{\Gamma(s)\Gamma((1-s)/2)}{\Gamma(s/2)\Gamma(1-s)}\right)^n. \tag{5}$$

We now define the higher relative class group, a group whose order is determined by values of $L(\chi,s)$ at negative even integers—see Kolster [8, 9] for a more complete treatment. Let $m\in\mathbb{Z}_{\geq 3}$ be odd. For a prime p, we denote by $H^i_{\text{ét}}(\mathbb{Z}_E[1/p],\mathbb{Z}_p(m))$ the ith étale cohomology group of $\operatorname{Spec}\mathbb{Z}_E[1/p]$ with m-fold twisted $\mathbb{Z}_p$-coefficients. Define the (mth) *higher class group* of E by

$$H^2(\mathbb{Z}_E,\mathbb{Z}(m))=\prod_p H^2_{\text{ét}}(\mathbb{Z}_E[1/p],\mathbb{Z}_p(m)).$$

There exists a homomorphism $K_{2m-2}(\mathbb{Z}_E)\to H^2(\mathbb{Z}_E,\mathbb{Z}(m))$ with finite cokernel [5, 18] (in fact, supported at 2), and it follows that $H^2(\mathbb{Z}_E,\mathbb{Z}(m))$ is finite, since $K_{2m-2}(\mathbb{Z}_E)$ is finite (a result of Borel [2] and Quillen [16]). We let $h_m(E)=\#H^2(\mathbb{Z}_E,\mathbb{Z}(m))$ denote the (mth) *higher class number* of E. We again have the divisibility $h_m(F)\mid h_m(K)$, and we define the *higher relative class number* of the CM extension K/F to be the quotient

$$h_m^-(K)=\frac{h_m(K)}{h_m(F)}\in\mathbb{Z}.$$

Two other quantities appear in the higher class number formula. First, let

$$H^0(E,\mathbb{Q}/\mathbb{Z}(m))=\prod_p H^0(E,\mathbb{Q}_p/\mathbb{Z}_p(m))$$

be the group of *higher roots of unity*, given in terms of Galois cohomology (invariants), and let $w_m(E)=\#H^0(E,\mathbb{Q}/\mathbb{Z}(m))$ denote its order. By definition, $w_m(E)$ is the largest integer q such that $G=\mathrm{Gal}(E(\zeta_q)/E)$ has exponent dividing m, where ζ_q denotes a primitive qth root of unity.

We have the following lemma that characterizes $w_m(E)$. If q is the power of a prime, let $\mathbb{Q}(\zeta_q)^{(m)}$ denote the subfield of $\mathbb{Q}(\zeta_q)$ of index $\gcd(\phi(q),m)$.

Lemma 1.1. *If q is a power of a prime, then $q\mid w_m(E)$ if and only if E contains $\mathbb{Q}(\zeta_q)^{(m)}$.*

Proof. Note that since $\mathbb{Q}(\zeta_q)$ is Galois, we have

$$G=\mathrm{Gal}(E(\zeta_q)/E)\cong\mathrm{Gal}(\mathbb{Q}(\zeta_q)/(E\cap\mathbb{Q}(\zeta_q)))\subset\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})\cong(\mathbb{Z}/q\mathbb{Z})^\times.$$

Since $m\geq 3$ is odd, if q is even then $q\mid w_m(E)$ if and only if $\mathbb{Q}(\zeta_q)\subset E$, as claimed. If q is odd, then since $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ is cyclic, we have $q\mid w_m(E)$ if and only if G has order dividing m if and only if $\mathbb{Q}(\zeta_q)^{(m)}\subset E$. $\square$

Finally, we define the *higher Q-unit index* of the CM extension K/F by

$$Q_m(K) = [H^1_{\text{ét}}(K, \mathbb{Z}_2(m)) : H^1_{\text{ét}}(F, \mathbb{Z}_2(m)) H^0(K, \mathbb{Q}_2/\mathbb{Z}_2(m))].$$

Collecting results of Kolster [9, §3], we know the following facts about Q_m.

Proposition 1.2.

(a) $Q_m \in \{1, 2\}$.
(b) $Q_m = 2$ *if and only if* $H^1(K/F, H^1_{\text{ét}}(K, \mathbb{Z}_2(m))) = 0$.
(c) *If an odd prime of* F *ramifies in* K, *then* $Q_m = 1$.
(d) *If no odd prime of* F *ramifies in* K, *the field* F *has only one prime lying above* 2, *and* $h(F)$ *is odd, then* $Q_m = 2$.

With these definitions, we now state the higher relative class number formula.

Proposition 1.3 (Kolster [9])**.** *We have*

$$L(\chi, 1-m) = (-1)^{n\left(\frac{m-1}{2}\right)} \frac{2^{n+1}}{Q_m(K)} \frac{h^-_m(K)}{w_m(K)} \tag{6}$$

up to a power of 2. *If the* 2*-adic Iwasawa main conjecture holds, then* (6) *holds.*

The proof of the Iwasawa main conjecture for p odd by Wiles [25] implies that the formula (6) holds up to a power of 2, and a proof of the 2-adic main conjecture would imply it exactly. Indeed, Kolster [9] has proven that (6) holds in the case that K is abelian over $\mathbb{Q}$ by applying Wiles' proof of the 2-adic main conjecture when K is abelian.

2. Estimates

In this section, we estimate the size of the relative class number. Throughout, we assume the truth of the higher relative class number formula (Proposition 1.3). We retain the notation from the previous section, suppressing the dependence on K whenever possible; in particular, we recall that $m \in \mathbb{Z}_{\geq 3}$ is odd.

We begin by substituting $s = m$ into (5). From standard Γ-function identities, letting $m = 2k - 1$ we have

$$\Gamma(m/2) = \frac{(2k)!}{4^k k!} \sqrt{\pi}$$

and

$$\frac{\Gamma((1-m)/2)}{\Gamma(1-m)} = \frac{\Gamma(-k)}{\Gamma(-2k)} = (-1)^k \frac{(2k)!}{k!}.$$

Putting these together, we obtain

$$\frac{\Gamma(m)\Gamma((1-m)/2)}{\Gamma(m/2)\Gamma(1-m)} = (-1)^{(m-1)/2} \frac{2^m (m-1)!}{\sqrt{\pi}}$$

hence

$$L(\chi, 1-m) = (-1)^{n\left(\frac{m-1}{2}\right)} L(\chi, m) \left(\frac{d_K}{d_F}\right)^{m-1/2} \left(\frac{2(m-1)!}{(2\pi)^m}\right)^n. \tag{7}$$

Now, by the higher relative class number formula, we have

$$h^-_m = |L(\chi, 1-m)| \frac{w_m Q_m}{2^{n+1}} \geq \frac{|L(\chi, 1-m)|}{2^{n+1}} \tag{8}$$

since $Q_m, w_m \in \mathbb{Z}_{\geq 1}$. From (7) and (8) we obtain

$$h_m^- \geq \frac{1}{2} L(\chi, m) \left(\frac{d_K}{d_F}\right)^{m-1/2} \left(\frac{(m-1)!}{(2\pi)^m}\right)^n. \tag{9}$$

We now estimate the value of $L(\chi, m)$.

Lemma 2.1. *We have $\gamma(m)^n \leq L(\chi, m) \leq \zeta(m)^n$, where*

$$\gamma(m) = \prod_p \left(1 + \frac{1}{p^m}\right)^{-1}.$$

Proof. From (4) we obtain

$$L(\chi, m) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^m}\right)^{-1}.$$

Organizing the product by primes p, we have

$$\left(1 + \frac{1}{p^m}\right)^{-n} \leq \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^m}\right)^{-1} \leq \left(1 - \frac{1}{p^m}\right)^{-n}$$

hence

$$\gamma(m)^n \leq L(\chi, m) \leq \zeta(m)^n$$

as claimed. □

We compute easily that $\gamma(3) \geq 0.8463$, $\gamma(5) \geq 0.9653$, and $\gamma(m) \geq 0.9917$ if $m \geq 7$; clearly $\gamma(m) \to 1$ as $m \to \infty$. Applying Lemma 2.1 to (9) we obtain

$$h_m^- \geq \frac{1}{2} \left(\frac{d_K}{d_F}\right)^{m-1/2} \left(\gamma(m) \frac{(m-1)!}{(2\pi)^m}\right)^n. \tag{10}$$

We pause to prove the following proposition, which is in the spirit of the Brauer-Siegel theorem. We have $d_K/d_F = d_F N\mathfrak{d}_{K/F} \geq d_F$, where $\mathfrak{d}_{K/F} \subset \mathbb{Z}_F$ denotes the relative discriminant of K/F.

Proposition 2.2. *Let $\{K_i/F_i\}_i$ be a sequence of CM extensions with*

$$[F_i : \mathbb{Q}] = o\left(\log(d_{K_i}/d_{F_i})\right).$$

Then

$$\log h_m^-(K_i) \sim (m - 1/2) \log(d_{K_i}/d_{F_i})$$

as $i \to \infty$.

Proof. Let K/F be a CM extension with $[F : \mathbb{Q}] = n$. From (10) we have

$$\log h_m^- \geq (m - 1/2) \log(d_K/d_F) + nc(m) \tag{11}$$

where $c(m)$ is a constant depending only on m. On the other hand, by Lemma 2.1 we have $L(\chi, m) \leq \zeta(m)^n$. Then applying equations (7) and (8) with this estimate we obtain

$$\log h_m^- \leq \log w_m + (m - 1/2) \log(d_K/d_F) + n \log \zeta(m). \tag{12}$$

Putting together (11) and (12), since $[F_i : \mathbb{Q}] = o(\log(d_{K_i}/d_{F_i}))$ as $i \to \infty$, the result follows if we show that

$$\log w_m(K_i) = o(\log(d_{K_i}/d_{F_i})). \tag{13}$$

To prove (13), we compare $w_m(K)$ and d_K/d_F for a CM extension K/F. Let $q = p^r$ be a power of a prime and suppose that $q \mid w_m$. By Lemma 1.1, the field K must contain $\mathbb{Q}(\zeta_q)^{(m)}$. Suppose first that q is odd. It follows that F must contain the field $\mathbb{Q}(\zeta_q)^{(2m)}$; since $d_{\mathbb{Q}(\zeta_q)^{(2m)}} \mid d_F$, by the conductor-discriminant formula [23, Theorem 3.11] we have

$$p^{\phi(q)/(2m)-1} \mid d_{\mathbb{Q}(\zeta_q)^{(2m)}}.$$

If q is even, then F must contain the totally real subfield $\mathbb{Q}(\zeta_q)^+$ of $\mathbb{Q}(\zeta_q)$, and we similarly conclude that $p^{\phi(q)/2-1} \mid d_F$. Define the multiplicative function f with the value $f(p^r) = p^{\phi(p^r)/(2m)-1}$ for a prime power p^r.

Now let K_i/F_i be a subsequence with $w_m(K_i) \to \infty$; if no such subsequence exists, then we are done. We now show that for any sequence of positive integers $n_i \to \infty$, we have $\log n_i = o(\log f(n_i))$. The result then follows as

$$\log w_{m_i} = o(\log f(w_{m_i})) = o(\log d_{F_i}) = o(\log(d_{K_i}/d_{F_i}))$$

as desired.

So let $\epsilon > 0$. We prove that

$$r < \epsilon\left(\frac{\phi(p^r)}{2m} - 1\right)$$

for all sufficiently large prime powers p^r, or also it is sufficient to show that

$$\frac{\phi(p^r)}{4rm} > \frac{1}{\epsilon}. \tag{14}$$

For $r = 1$, clearly the inequality (14) will be satisfied for p sufficiently large; but then for the finitely many remaining primes p, the inequality holds for a sufficiently large power r. Therefore it holds for any sufficiently large prime power p^r, and claimed. ☐

Corollary 2.3. *Let F be a totally real field and $m \in \mathbb{Z}_{\geq 3}$ be odd. Then for any $h \in \mathbb{Z}_{\geq 1}$, there are only finitely many CM extensions K/F with $h_m^-(K) \leq h$.*

We return to estimating h_m^- from below. Rewriting (10) we obtain

$$d_F \leq \frac{d_K}{d_F} \leq (2h_m^-)^{1/(m-1/2)} C(m)^n \tag{15}$$

where

$$C(m) = \left(\frac{(2\pi)^m}{\gamma(m)(m-1)!}\right)^{1/(m-1/2)}$$

depends only on m. We compute that

$$C(3) \leq 7.3517, \quad C(5) \leq 3.8332, \quad C(7) \leq 2.6336, \quad C(9) \leq 2.011.$$

It follows from Stirling's approximation and the fact that $\gamma(m)$ is increasing to 1 that $C(m)$ is decreasing to 0.

Let $\delta_F = d_F^{1/n}$ denote the root discriminant of F. Taking nth roots we obtain from (15) that

$$\delta_F \leq (2h_m^-)^{2/((2m-1)n)} C(m). \tag{16}$$

3. Enumerating the list of extensions

We now apply the results of §2 to enumerate CM extensions with small higher relative class number.

We list those extensions with $h_m^- \leq 16$; we have chosen this bound to capture the smallest higher relative class number of a nonabelian field, and we note that this bound can easily be increased, if desired. Let $NF_m(n)$ denote the set of totally real fields F of degree n that satisfy the bound (16) with $h_m^- = 16$, and let $NF_m = \bigcup_n NF_m(n)$.

For $m = 3$, the estimate (16) then reads

$$\delta_F \leq 32^{2/(5n)} C(3) \leq 7.3517 \cdot 4^{1/n}. \tag{17}$$

By the (unconditional) Odlyzko bounds [14], if $n \geq 7$ we have $\delta_F \geq 9.301$, but by (17) we have $\delta_F \leq 8.962$, a contradiction. In Table 2.3, for each degree $n \geq 2$ we list the upper bound on the root discriminant δ_F, the corresponding Odlyzko bound B_O, the corresponding upper bound on the discriminant d_F, and the size of $NF_3(n)$.

Table 2.3. Degree and Root Discriminant Bounds

n	2	3	4	5	6	≥ 7
δ_F	≤ 14.703	11.670	10.397	9.701	9.263	8.962
B_O	> 2.223	3.610	5.067	6.523	7.941	9.301
d_F	≤ 216	1589	11684	85899	631505	–
$\#NF_3(n)$	65	48	64	8	6	0

The fields $NF_3(n)$ of such small discriminant are well-known [22].

For $m = 5$, arguing in a similar way we have

$$\delta_F \leq 32^{2/(9n)} C(5) \leq 3.8332 \cdot 2.1602^{1/n},$$

which already for $n \geq 4$ gives $\delta_F \leq 4.5273$, contradicting the Odlyzko bound $\delta_F > 5.067$. In fact, we have $d_F \leq 28$ for $n = 2$ and $d_F \leq 109$ for $n = 3$, so that $\#NF_5(2) = 8$ and $\#NF_5(3) = 2$. Proceeding in this way, we find:

$$NF_m = \begin{cases} \{\mathbb{Q}, \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{8})\}, & \text{if } m = 7; \\ \{\mathbb{Q}, \mathbb{Q}(\sqrt{5})\}, & \text{if } m = 9; \\ \{\mathbb{Q}\}, & \text{if } 11 \leq m \leq 19; \\ \emptyset, & \text{if } m \geq 21. \end{cases}$$

Now for each such field F and m, we have from (15) that

$$d_K \leq \lfloor 32^{2/(2m-1)} C(m)^n \rfloor d_F, \tag{18}$$

leaving only finitely many possibilities for the CM extension K/F. We can find these relative quadratic extensions explicitly by using a relative version of Hunter's theorem due to Martinet: see Cohen [4, §§9.2–9.3] for more details. We obtain in this way $90, 9, 2, 1$ extensions K/F for $m = 3, 5, 7, 9$, and none for $m \geq 11$.

Next, using (7) and the higher relative class number formula (Proposition 1.3) we numerically compute the value $h_m^-/(w_m Q_m) \in \mathbb{R}$. In order to recover this value exactly, it suffices to bound the size of the denominator. We have $Q_m \in \{1, 2\}$.

To determine w_m, we apply Lemma 1.1. If q is the power of a prime and $q \mid m$, then q is odd and K contains the unique subfield of index m of $\mathbb{Q}(\zeta_q)$. In particular, this implies that $\phi(q)/m \mid 2n = [K:\mathbb{Q}]$, which already gives a bound on w_m. To reduce the size of this bound further, we note that we also have $d_{\mathbb{Q}(\zeta_q)^{(m)}} \mid d_K$, so in particular $q \mid d_K$ whenever $\phi(q) > m$; furthermore, for any prime $\mathfrak{p}$ of K that is prime to qd_K, it is easy to see that the order of $N\mathfrak{p} \in (\mathbb{Z}/q\mathbb{Z})^*$ must divide m. A prime power q that passes these tests, the latter for sufficiently many primes $\mathfrak{p}$, is very likely to divide w_m. To compute w_m exactly and verify that indeed $q \mid w_m$, we simply check if the qth cyclotomic polynomial over K factors into polynomials of degree at most m.

In this way, we compute $h_m^-/Q_m \in \frac{1}{2}\mathbb{Z}$. If this value is not an integer, then we immediately know $Q_m = 2$. Otherwise, we may apply the tests of Proposition 1.2 to determine the value Q_m in almost all cases. We are lucky that in many cases where we cannot determine the value of Q_m, we nevertheless have $h_m^- \geq h_m^-/Q_m > 16$. We have 6 remaining cases. One of which we can resolve as follows: for $F = \mathbb{Q}(\sqrt{30})$ and $K = \mathbb{Q}(\sqrt{-15}, \sqrt{-2})$ with $h_3^-/Q_3 = 12$, we apply the divisibility result $8 = h_3^-(\mathbb{Q}(\sqrt{-15})) \mid h_3^-(K)$ [9, Corollary 3.6], which implies $Q_3 = 2$, and hence $h_3^- = 24$. We were unable to resolve, and we leave it an open problem to compute the higher Q-index Q_m in the other 5 cases. We expect the problem to be nontrivial for the reason that already characterizing Hasse's Q-index is quite intricate (see Hasse [7]).

Of the $90, 9, 2, 1$ CM extensions for $m = 3, 5, 7, 9$, respectively, $26 - 29, 4, 1, 0$ have $h_m^- \leq 16$, and they are listed in Tables 4.1–4.2.

4. Tables

In this section, we present the tables of CM extensions with higher relative class number $h_m^- \leq 16$. Below, we list the totally real field F and its discriminant d_F, the CM field K, its absolute discriminant d_K and the norm of the relative discriminant $N(\mathfrak{d}_{K/F})$, an element $\delta \in F$ such that $K = F(\sqrt{\delta})$, and the higher class number h_m^-. As usual, we let ζ_k denote a primitive kth root of unity, $\omega = \zeta_3$ and $i = \zeta_4$, and we define $\lambda_k = \zeta_k + 1/\zeta_k$, so that $\mathbb{Q}(\lambda_k) = \mathbb{Q}(\zeta_k)^+$.

The computations were performed in Magma [13] and Sage [17]; total computing time was about 2 minutes.

It is interesting to note that there is no CM extension with higher relative class number $h_m^- = 2$. Also, note that there is a misprint in the computation of $h_m^-(\zeta_{12}) = 1$ in Kolster [9].

Table 4.1. CM extensions with higher relative class number $h_m^- \leq 16$ for $m \geq 5$

d_F	F	d_K	K	$N(\mathfrak{d}_{K/F})$	δ	h_5^-
1	$\mathbb{Q}$	3	$\mathbb{Q}(\omega)$	3	-3	1
1	$\mathbb{Q}$	4	$\mathbb{Q}(i)$	4	-4	5
12	$\mathbb{Q}(\sqrt{3})$	144	$\mathbb{Q}(\zeta_{12})$	1	-1	5
1	$\mathbb{Q}$	7	$\mathbb{Q}(\sqrt{-7})$	7	-7	16

d_F	F	d_K	K	$N(\mathfrak{d}_{K/F})$	a	h_7^-
1	$\mathbb{Q}$	3	$\mathbb{Q}(\omega)$	3	-3	7

Table 4.2. CM extensions with higher relative class number $h_3^- \le 16$

d_F	F	d_K	K	$N(\mathfrak{d}_{K/F})$	δ	h_3^-
1	$\mathbb{Q}$	3	$\mathbb{Q}(\omega)$	3	-3	1
1	$\mathbb{Q}$	4	$\mathbb{Q}(i)$	4	-1	1
5	$\mathbb{Q}(\sqrt{5})$	125	$\mathbb{Q}(\zeta_5)$	5	$-\sqrt{5}(1+\sqrt{5})/2$	1
12	$\mathbb{Q}(\sqrt{3})$	144	$\mathbb{Q}(\zeta_{12})$	1	-1	1
1	$\mathbb{Q}$	8	$\mathbb{Q}(\sqrt{-2})$	8	-2	3
1	$\mathbb{Q}$	11	$\mathbb{Q}(\sqrt{-11})$	11	-11	3
8	$\mathbb{Q}(\sqrt{2})$	256	$\mathbb{Q}(\zeta_8)$	4	-2	3
24	$\mathbb{Q}(\sqrt{6})$	576	$\mathbb{Q}(\sqrt{-2},\omega)$	1	-2	3
44	$\mathbb{Q}(\sqrt{11})$	1936	$\mathbb{Q}(\sqrt{-11},i)$	1	-1	3
33	$\mathbb{Q}(\sqrt{33})$	1089	$\mathbb{Q}(\sqrt{-11},\omega)$	1	-3	3 or 6
60	$\mathbb{Q}(\sqrt{15})$	3600	$\mathbb{Q}(\sqrt{-15},i)$	1	-1	4 or 8
1	$\mathbb{Q}$	7	$\mathbb{Q}(\sqrt{-7})$	7	-7	8
1	$\mathbb{Q}$	15	$\mathbb{Q}(\sqrt{-15})$	15	-15	8
5	$\mathbb{Q}(\sqrt{5})$	225	$\mathbb{Q}(\sqrt{-15},\omega)$	9	-3	8
28	$\mathbb{Q}(\sqrt{7})$	784	$\mathbb{Q}(\sqrt{-7},i)$	1	-1	8
49	$\mathbb{Q}(\lambda_7)$	16807	$\mathbb{Q}(\zeta_7)$	7	-7	8
88	$\mathbb{Q}(\sqrt{22})$	7744	$\mathbb{Q}(\sqrt{-11},\sqrt{-2})$	16	-2	9
1	$\mathbb{Q}$	19	$\mathbb{Q}(\sqrt{-19})$	19	-19	11
57	$\mathbb{Q}(\sqrt{57})$	3249	$\mathbb{Q}(\sqrt{-19},\omega)$	1	-3	11 or 22
76	$\mathbb{Q}(\sqrt{19})$	5776	$\mathbb{Q}(\sqrt{-19},i)$	1	-1	11
81	$\mathbb{Q}(\lambda_9)$	19683	$\mathbb{Q}(\zeta_9)$	3	-3	13
1	$\mathbb{Q}$	20	$\mathbb{Q}(\sqrt{-5})$	20	-5	15
5	$\mathbb{Q}(\sqrt{5})$	400	$\mathbb{Q}(\sqrt{-5},i)$	16	-1	15
60	$\mathbb{Q}(\sqrt{15})$	3600	$\mathbb{Q}(\sqrt{-5},\omega)$	1	-1	15 or 30
2000	$\mathbb{Q}(\lambda_{20})$	4000000	$\mathbb{Q}(\zeta_{20})$	1	-1	15
21	$\mathbb{Q}(\sqrt{21})$	441	$\mathbb{Q}(\sqrt{-7},\omega)$	1	-3	16
8	$\mathbb{Q}(\sqrt{2})$	1088	$\mathbb{Q}\left(\sqrt{2\sqrt{2}-5}\right)$	17	$2\sqrt{2}-5$	16
17	$\mathbb{Q}(\sqrt{17})$	2312	$\mathbb{Q}\left(\sqrt{-(5+\sqrt{17})/2}\right)$	8	$-(5+\sqrt{17})/2$	16 or 32
1125	$\mathbb{Q}(\lambda_{15})$	1265625	$\mathbb{Q}(\zeta_{15})$	1	-15	16

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