ON GALOIS INERTIAL TYPES OF ELLIPTIC CURVES OVER $\mathbb{Q}_p$

LASSINA DEMBÉLÉ, NUNO FREITAS, AND JOHN VOIGHT

Abstract. We provide a complete, explicit description of the inertial Weil–Deligne types arising from elliptic curves over $\mathbb{Q}_p$ for p prime.

1. Introduction

1.1. **Motivation.** In the classification of finite-dimensional complex representations of the absolute Galois group of a local field, it has proven to be very useful to classify by restriction to the inertia subgroup [10, 33, 9, 29]. In this article, we will pursue an explicit classification for such representations coming from elliptic curves.

Let p be prime and let $F \supseteq \mathbb{Q}_p$ be a finite extension with algebraic closure F^{al}. Let W_F be the Weil group of F, the subgroup of $\mathrm{Gal}(F^{\mathrm{al}} \,|\, F)$ acting by an integer power of the Frobenius map on the maximal unramified subextension. Let $(\rho\colon W_F \to \mathrm{GL}_n(\mathbb{C}), N)$ be an n-dimensional (complex) Weil–Deligne representation (Definition 2.1.3). Let $I_F \leq W_F$ be the inertia subgroup. An inertial (Weil–Deligne) type (also called a Galois inertial type) is a pair (τ, N) where $\tau = \rho|_{I_F}$ for a Weil–Deligne representation (ρ, N). To ease notation, we will often abbreviate the pair (τ, N) by τ (and indeed often we have $N = 0$ anyway).

Already the case $n = 2$ is interesting and rich, and we will consider this case here. Inertial types for 2-dimensional representations were introduced by Conrad–Diamond–Taylor [14] and Breuil–Conrad–Diamond–Taylor [6] in the study of deformation rings of Galois representations and were used in the proof of modularity of elliptic curves over $\mathbb{Q}$. Diamond–Kramer [20, Appendix] described the analogously defined type (as in Diamond [19]) of the mod p Galois representation $\overline{\rho}_{E,p}$ attached to an elliptic curve E over F in terms of the j-invariant of E; in particular, they give a description of the restriction of $\overline{\rho}_{E,p}$ to I_F in as much detail as possible using only $j(E)$.

Types have also been studied in the context of Galois representations attached more generally to classical modular forms. For example, in Loeffler–Weinstein [30, 31] an algorithm to determine the restriction to decomposition groups of such representations was described and implemented; this includes a description of the inertial type. By counting the number of inertial types attached to modular forms, Dieulefait–Pacetti–Tsaknias [21] have given a precise generalization of the Maeda conjecture.

Additionally, inertial types have played a prominent role in the mod p and p-adic Langlands program. Henniart [7, Appendix] showed that there is an *inertial Langlands correspondence* between 2-dimensional Galois inertial types of F and smooth representations of $\mathrm{GL}_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ denotes the ring of integers of F. Indeed, the Breuil–Mézard conjecture [7] for $\mathbb{Q}_p$ can be seen as a refinement of Serre's conjecture over $\mathbb{Q}$, where inertial types are a crucial input. An inertial Langlands correspondence for general $n \geq 2$ was proven by Paškūnas [33].

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Diophantine applications provide another important motivation to study inertial types for GL_2. In Bennett–Skinner [3], the *image of inertia argument* was introduced and successfully applied to solve certain Fermat equations. Recently, further refinements and applications of this argument were obtained by Billerey–Chen–Dieulefait–Freitas [4]: we may be able to distinguish between the mod p representations attached to elliptic curves over a global field by showing they have different images of inertia [4, Section 3]. Therefore, the more we know about inertial types of elliptic curves, the greater the applicability of this argument. In this direction, Freitas–Naskręcki–Stoll [25, Theorem 3.1] describe the possible fixed fields of the restriction $\overline{\rho}_{E,p}|_{I_{\mathbb{Q}_p}}$ to inertia for elliptic curves E over $\mathbb{Q}_p$ with certain reduction types at $p = 2, 3$, and they applied this to study solutions of the generalized Fermat equation $x^2 + y^3 = z^p$.

In light of these applications, the goal of this paper is to give a complete, explicit description of the inertial types for all elliptic curves E over $\mathbb{Q}_p$. Our main theorem (Theorem 1.2 below) has already been applied to the determination of the symplectic type of isomorphisms between the p-torsion of elliptic curve by Freitas–Kraus [26].

1.2. **Main result.** Let E be an elliptic curve over $F = \mathbb{Q}_p$. Attached to E is an inertial Weil–Deligne type τ_E obtained from the action on the (dual of the) ℓ-adic Tate module for a prime $\ell \neq p$, independent of ℓ (for details, see Section 3.1). If E has potentially good reduction, then this good reduction is obtained over a minimal finite extension $L \supseteq F^{\mathrm{un}}$ where F^{un} denotes the maximal unramified extension of F, and we define the semistability defect of E to be $e_E := [L : F^{\mathrm{un}}]$.

Our main result (combining Lemma 3.2.4, Propositions 4.1.1, 4.2.1, and 5.2.2, and Theorems 6.1.4 and 7.1.2) is as follows.

Main Theorem. *Let E be an elliptic curve over $\mathbb{Q}_p$ with conductor N_E and inertial Weil–Deligne type τ_E; if E has additive, potentially good reduction, let e_E be its semistability defect. Then τ_E is classified up to equivalence according to Table 1.*

The notation in Table 1 is explained in Section 2.5. In Table 1, the exceptional (or primitive) supercuspidal representations are labelled as such and collected in the last rows of the table, whereas the nonexceptional (imprimitive) supercuspidal representations are labelled simply *supercuspidal*, for brevity.

All types in Table 1 arise for an elliptic curve over $\mathbb{Q}_p$: see Tables 4, 6.7, and 17. See Dokchitser–Dokchitser [23] for computation of Kodaira types—the type restricts the possible Kodaira types but not necessarily uniquely, so for simplicity we do not include them in our table.

Our method of proof of the Main Theorem is by direct calculation: we deduce the inertial type associated to an elliptic curve over $\mathbb{Q}_p$ in terms of its reduction type. We have endeavored to streamline these calculations while still remaining comprehensive and as self-contained as possible. Many of our calculations are performed in the computer algebra system Magma [5]; the code is available online [18].

Indeed, many of these calculations can be found in other places in the literature: for example, the 3-adic types are already implicitly given in the proof of the modularity theorem [6], and Dieulefait–Pacetti–Tsaknias [21] more generally identify local invariants of Galois orbits of classical newforms (relevant here for weight $k = 2$). Coppola [15, 16] has recently studied wild Galois representations ($p = 3$ and $e = 12$; $p = 2$ and $e = 8, 24$) over more general

Reduction type	p	e_E	$v_p(N_E)$	τ_E	Description
good	-	-	0	τ_{triv}	trivial
multiplicative	-	-	1	$\tau_{\mathrm{St},p}$	special
additive, potentially multiplicative	≥ 3	-	2	$\tau_{\mathrm{St},p} \otimes \varepsilon_p$	special
	2	-	4	$\tau_{\mathrm{St},2} \otimes \varepsilon_{-4}$	
			6	$\tau_{\mathrm{St},2} \otimes \varepsilon_{\pm 8}$	
additive, potentially good	≥ 5	2	2	$\tau_{\mathrm{triv}} \otimes \varepsilon_p$	principal series
		$3,4,6 \mid (p-1)$		$\tau_{\mathrm{ps},p}(1,1,e)$	
		$3,4,6 \mid (p+1)$		$\tau_{\mathrm{sc},p}(u,1,e)$	supercuspidal
	3	2	2	$\tau_{\mathrm{triv}} \otimes \varepsilon_3$	principal series
		3	4	$\tau_{\mathrm{ps},3}(1,2,3)$	
		3	4	$\tau_{\mathrm{sc},3}(-1,2,3)$	supercuspidal
		4	2	$\tau_{\mathrm{sc},3}(-1,1,4)$	
		6	4	$\tau_{\mathrm{ps},3}(1,2,3) \otimes \varepsilon_3$	principal series
		6	4	$\tau_{\mathrm{sc},3}(-1,2,3) \otimes \varepsilon_3$	supercuspidal
		12	3	$\tau_{\mathrm{sc},3}(\pm 3,2,6)$	
			5	$\tau_{\mathrm{sc},3}(-3,4,6)_j \quad (j=0,1,2)$	
	2	2	4	$\tau_{\mathrm{triv}} \otimes \varepsilon_{-4}$	principal series
			6	$\tau_{\mathrm{triv}} \otimes \varepsilon_{\pm 8}$	
		3	2	$\tau_{\mathrm{sc},2}(5,1,3)$	supercuspidal
		4	8	$\tau_{\mathrm{ps},2}(1,4,4) \otimes \varepsilon_d \quad (d=1,-4)$	principal series
				$\tau_{\mathrm{sc},2}(5,4,4) \otimes \varepsilon_d \quad (d=1,-4)$	supercuspidal
		6	4	$\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$	
			6	$\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{\pm 8}$	
		8	5	$\tau_{\mathrm{sc},2}(-4,3,4)$, $\tau_{\mathrm{sc},2}(-20,3,4)$	
			6	$\tau_{\mathrm{sc},2}(-4,3,4) \otimes \varepsilon_8$, $\tau_{\mathrm{sc},2}(-20,3,4) \otimes \varepsilon_8$	
			8	$\tau_{\mathrm{sc},2}(-4,6,4) \otimes \varepsilon_d \quad (d=1,-4)$	
		24	3	$\tau_{\mathrm{ex},2,1}$	exceptional supercuspidal
			4	$\tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-4}$	
			6	$\tau_{\mathrm{ex},2,1} \otimes \varepsilon_{\pm 8}$	
			7	$\tau_{\mathrm{ex},2,2} \otimes \varepsilon_d \ (d=1,-4,\pm 8)$	

TABLE 1. Inertial WD-types for elliptic curves over $\mathbb{Q}_p$

local fields, classifying the Galois representation up to isomorphism (as an abstract group) but without providing the explicit description of the underlying field. Finally, Barrios–Roy [1] recently studied representations (including the inertial type) for elliptic curves with nontrivial odd torsion.

Our proof has an interesting algorithmic consequence: to compute the type of an elliptic curve E over $\mathbb{Q}_p$, it suffices to run Tate's algorithm on E over an explicit finite list of extensions of $\mathbb{Q}_p$, see e.g. Corollary 5.3.4. For reliability, we have implemented this algorithm on over thousands of elliptic curves over $\mathbb{Q}$; the code is available online [18]. Dokchitser–Dokchitser also implemented algorithms for working with the Galois representations attached to elliptic curves over local fields in Magma [5] by reconstructing representations from their (good) Euler factors [24] and a tame local reciprocity formula of Newton [32]. This implementation does not compute the inertia field (necessary to distinguish the types). Our method of listing types can also be used to algorithmically enumerate all 2-dimensional inertial types by increasing conductor; however, most of the effort in this paper is spent identifying which types arise from elliptic curves.

1.3. **Outline.** The paper is organized as follows. In sections 2–3, we establish background by briefly reviewing some facts concerning 2-dimensional Weil–Deligne representations, inertial types, and elliptic curves. In Section 4, we compute types in the case of potentially multiplicative reduction for all primes p and for additive, potentially good reduction for $p \geq 5$. The remainder of the paper is concerned with additive, potentially good reduction first for $p = 3$ (Section 5) then $p = 2$ (sections 6–7 for the nonexceptional and exceptional cases).

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2. Two-dimensional Weil–Deligne representations

In this section, we quickly recall background on Galois representations of local fields and types. Our main references are Tate [38], Rohrlich [34], Carayol [11], and Bushnell–Henniart [8].

2.1. **Notation.** A (complex) quasicharacter of a topological group G is a continuous homomorphism $\chi\colon G \to \mathbb{C}^\times$ with open kernel; if further $|\chi(g)| = 1$ for all $g \in G$, we call χ a (unitary) character. If $\chi\colon G \to \mathbb{C}^\times$ is a (quasi)character and $\varphi\colon G \to G'$ is a continuous group homomorphism, we say that χ factors through φ if there exists a (quasi)character $\chi'\colon G' \to \mathbb{C}^\times$ such that $\chi = \chi' \circ \varphi$.

Remark 2.1.1. Throughout, one can equally well replace $\mathbb{C}$ with any algebraically closed field of characteristic 0, for example an algebraic closure of $\mathbb{Q}_\ell$ with $\ell \neq p$.

Let p be prime and let $F \supseteq \mathbb{Q}_p$ be a finite extension with algebraic closure F^{al} and maximal unramified extension $F^{\mathrm{un}} \subset F^{\mathrm{al}}$. Let $\mathcal{O}_F \subset F$ be the valuation ring of F with maximal ideal $\mathfrak{p}$, uniformizer $\pi \in \mathfrak{p}$, and residue field k of cardinality $q := \#k$. Let $v\colon F^\times \to \mathbb{Z}$ denote the valuation of F normalized with $v(\pi) = 1$, and let $|\cdot|_v\colon F^\times \to \mathbb{R}_{>0}^\times$ be the associated normalized absolute value. Let $W_F < \mathrm{Gal}(F^{\mathrm{al}} \,|\, F)$ be the Weil group of F and $I_F < W_F$ its inertia subgroup, fitting into the exact sequence

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1. \tag{2.1.2}$$

For $F = \mathbb{Q}_p$, for brevity we replace F by p in the subscript, e.g., writing $I_p < W_p$.

Let W_F^{ab} denote the maximal abelian quotient of W_F and let $\mathrm{Art}_F\colon F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ be the Artin reciprocity map from local class field theory, the isomorphism of topological groups sending $\pi \in \mathcal{O}_F$ to the class of a *geometric* Frobenius element $\mathrm{Fr} \in W_F^{\mathrm{ab}}$ characterized by $\mathrm{Fr}(x^q) = x$ for $x \in k$. The map Art_F allows us to identify a (quasi)character χ of W_F with the (quasi)character $\chi^{\mathrm{A}} := \chi \circ \mathrm{Art}_F$ of $F^\times$, and conversely. The conductor of χ is the ideal $\mathrm{cond}(\chi) := \mathfrak{p}^m$ where $\mathrm{condexp}(\chi) := m \in \mathbb{Z}_{\geq 0}$ is conductor exponent, the smallest nonnegative integer such that the restriction $\chi^{\mathrm{A}}|_{1+\mathfrak{p}^m}$ to $1 + \mathfrak{p}^m \leq \mathcal{O}_K^\times$ is trivial. Let $\omega\colon W_F \to \mathbb{C}^\times$ be the quasicharacter corresponding to the norm quasicharacter $|\cdot|_v$, so that $\omega(g) = q^{-a}$ for $g|_{F^{\mathrm{un}}} = \mathrm{Fr}^a$ with $a \in \mathbb{Z}$.

Definition 2.1.3. A (n-dimensional) Weil–Deligne representation is a pair (ρ, N) such that:

(i) $\rho\colon W_F \to \mathrm{GL}_n(\mathbb{C})$ is a homomorphism with open kernel; and
(ii) $N \in \mathrm{GL}_n(\mathbb{C})$ is nilpotent and satisfies

$$\rho(g) N \rho(g)^{-1} = \omega(g) N \quad \text{for all } g \in W_F. \tag{2.1.4}$$

An isomorphism (or equivalence) of Weil–Deligne representations from (ρ, N) to (ρ', N') is specified by an element $P \in \mathrm{GL}_n(\mathbb{C})$ such that $\rho'(g) = P\rho(g)P^{-1}$ for all $g \in W_F$ and $N' = PNP^{-1}$.

Remark 2.1.5. The nilpotent element N comes from the fact that there exists an open subgroup $H \leq I_F$ such that $\rho|_H$ is unipotent; below, we will usually have $N = 0$.

2.2. **Classification.** Every 2-dimensional Weil–Deligne representation arises up to isomorphism from one of the following three possibilities.

- *Principal series.* Let $\chi_1, \chi_2\colon W_F \to \mathbb{C}^\times$ be quasicharacters such that $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$. The principal series representation associated to χ_1, χ_2 is $(\mathrm{PS}(\chi_1, \chi_2), 0)$, where

$$\mathrm{PS}(\chi_1, \chi_2) := \chi_1 \oplus \chi_2.$$

  Its conductor exponent is given by

$$\mathrm{condexp}(\mathrm{PS}(\chi_1, \chi_2)) = \mathrm{condexp}(\chi_1) + \mathrm{condexp}(\chi_2). \tag{2.2.1}$$

- *Special or Steinberg representations.* Let $\chi\colon W_F \to \mathbb{C}^\times$ be a quasicharacter. The special or Steinberg representation associated to χ is $(\mathrm{St}(\chi), N)$, where

$$\mathrm{St}(\chi) := \chi\omega \oplus \chi \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.2.2}$$

  We have

$$\mathrm{condexp}(\mathrm{St}(\chi)) = \begin{cases} 2\,\mathrm{condexp}(\chi), & \text{if } \chi \text{ is ramified;} \\ 1, & \text{otherwise.} \end{cases} \tag{2.2.3}$$

- *Supercuspidal representations.* The Weil–Deligne representations $(\rho, 0)$ where ρ is an *irreducible* 2-dimensional representation of W_F are called supercuspidal. Supercuspidal representations are classified by their projective images in $\mathrm{PGL}_2(\mathbb{C})$ (see Bushnell–Henniart [8, sections 41 and 42] or Carayol [11, section 12]). We say that a supercuspidal ρ is nonexceptional if its projective image is dihedral, otherwise we say that ρ is exceptional; in the latter case ρ has projective image A_4 or S_4. (Since W_F is totally disconnected, the projective image A_5 cannot occur.) Alternatively, the nonexceptional representations are also called imprimitive and the exceptional representations are called primitive.

2.3. **Nonexceptional supercuspidal representations.** Since they will command significant attention here, we explore supercuspidal representations further. We begin with the nonexceptional supercuspidal representations.

Let $K \supset F$ be a quadratic extension and let $\psi_K \colon W_F \to \{\pm 1\}$ be the quadratic character of W_F with kernel W_K. Let $\chi \colon W_K \to \mathbb{C}^\times$ be a quasicharacter and consider the associated quasicharacter $\chi^{\mathrm{A}} \colon K^\times \to \mathbb{C}^\times$. Let $s \in W_F$ be a lift of the nontrivial element in $\mathrm{Gal}(K \mid F)$. Since $W_K \trianglelefteq W_F$ is normal (with s representing the nontrivial coset), the s-conjugate of χ

$$\begin{aligned} \chi^s \colon W_K &\to \mathbb{C}^\times \\ \chi^s(g) &= \chi(s^{-1} g s), \end{aligned} \tag{2.3.1}$$

is independent of the choice of s. By local class field theory, we have $(\chi^s)^{\mathrm{A}} = \chi^{\mathrm{A}} \circ s$.

Lemma 2.3.2. *The following are equivalent:*

(i) $\chi^s = \chi$*;*
(ii) $s(x)/x \in \ker \chi^{\mathrm{A}}$ *for all* $x \in K^\times$*; and*
(iii) χ^{A} *factors through the norm map* $\mathrm{Nm}_{K|F} \colon K^\times \to F^\times$*.*

Of course, since $K^\times$ is generated by π (for any choice of uniformizer π) and $\mathcal{O}_K^\times$, it is enough to check (ii) for $x = \pi$ and all $x \in \mathcal{O}_K^\times$.

Proof. For (i) $\Leftrightarrow$ (ii), we observe

$$\begin{aligned} \chi^s = \chi \quad &\Leftrightarrow \quad (\chi^s)^{\mathrm{A}}(x) = \chi^{\mathrm{A}}(x) \quad \text{for all } x \in K^\times \\ &\Leftrightarrow \quad \chi^{\mathrm{A}}(s(x)/x) = 1 \quad \text{for all } x \in K^\times \\ &\Leftrightarrow \quad \{s(x)/x : x \in K^\times\} \le \ker \chi^{\mathrm{A}}. \end{aligned} \tag{2.3.3}$$

For (ii) $\Leftrightarrow$ (iii), we recall that $\ker(\mathrm{Nm}_{K|F}) = \{s(x)/x : x \in K^\times\}$ by Hilbert's Theorem 90, so (ii) holds if and only if χ^{A} factors through the surjective norm map $\mathrm{Nm}_{K|F} \colon K^\times \to \mathrm{Nm}_{K|F}(K^\times)$. But of course $\mathrm{Nm}_{K|F}(K^\times) \le F^\times$ is open of finite index, so θ' extends to a quasicharacter $\theta \colon F^\times \to \mathbb{C}^\times$, and we see that (ii) $\Leftrightarrow$ (iii). $\square$

We record the following consequences.

Corollary 2.3.4. *The following statements hold.*

(a) *If* $\chi^{\mathrm{A}}(-1) = -1$ *then* χ^{A} *does not factor via the norm map.*
(b) *Suppose* $\mathcal{O}_F^\times \le \ker \chi^{\mathrm{A}}$*. Then* χ^{A} *factors through the norm map if and only if* $\chi^{\mathrm{A}}(s(\pi)/\pi) = 1$ *and* $\chi^{\mathrm{A}}|_{\mathcal{O}_K^\times}$ *is quadratic.*
(c) *The character* $(\chi^s/\chi)^{\mathrm{A}}$ *factors through the norm map if and only if* χ^s/χ *is quadratic.*

Proof. For (a), if $\chi^{\mathrm{A}} = \theta \circ \mathrm{Nm}_{K|F}$ then $\chi^{\mathrm{A}}(-1) = \theta((-1)^2) = 1$. For (b), we apply the equivalence (ii) $\Leftrightarrow$ (iii) of Lemma 2.3.2. For $u \in \mathcal{O}_K^\times$, by hypothesis we have

$$\chi^{\mathrm{A}}(s(u)/u) = \chi^{\mathrm{A}}(\mathrm{Nm}_{K|F}(u)/u^2) = 1/\chi^{\mathrm{A}}(u)^2$$

so $s(u)/u \in \ker \chi^{\mathrm{A}}$ if and only if $\chi^{\mathrm{A}}(u)^2 = 1$. Thus $s(x)/x \in \ker \chi^{\mathrm{A}}$ for all $x \in K^\times$ if and only if $\chi^{\mathrm{A}}(s(\pi)/\pi) = 1$ and $(\chi^{\mathrm{A}}|_{\mathcal{O}_K^\times})^2$ is trivial, i.e., $\chi^{\mathrm{A}}|_{\mathcal{O}_K^\times}$ is quadratic.

For (c), we apply the equivalence (i) $\Leftrightarrow$ (iii): so $(\chi^s/\chi)^{\mathrm{A}}$ factors through the norm map if and only if $(\chi^s/\chi)^s = \chi^s/\chi$. But $(\chi^s/\chi)^s = \chi/\chi^s = (\chi^s/\chi)^{-1}$, so the condition holds if and only if $\chi^s/\chi = (\chi^s/\chi)^{-1}$ if and only if χ^s/χ is quadratic. □

Suppose that $\chi \neq \chi^s$. Then the nonexceptional supercuspidal representation attached to χ is $(\mathrm{Ind}_{W_K}^{W_F} \chi, 0)$, the induction of χ from W_K to W_F (with $N = 0$). (The condition $\chi \neq \chi^s$ is necessary to ensure that $\mathrm{Ind}_{W_K}^{W_F} \chi$ is irreducible.) Recalling that ψ_K is the quadratic character of W_F corresponding to K, we have the following formula (see Serre [35, VI Proposition 4]):

$$\mathrm{condexp}(\mathrm{Ind}_{W_K}^{W_F} \chi) = \begin{cases} 2\,\mathrm{condexp}(\chi), & \text{if } K \mid F \text{ is unramified;} \\ \mathrm{condexp}(\chi) + \mathrm{condexp}(\psi_K), & \text{if } K \mid F \text{ is ramified.} \end{cases} \tag{2.3.5}$$

We conclude with a few simple lemmas.

Lemma 2.3.6. *If $\chi^s|_{I_K} = \chi^{-1}|_{I_K}$, then $(\chi^s/\chi)|_{I_K}$ has order dividing 2 if and only if $\chi|_{I_K}$ has order dividing 4.*

Proof. Indeed, we have $(\chi^s/\chi)|_{I_K} = \chi^{-2}|_{I_K}$. □

Lemma 2.3.7. *Let $\delta := \det(\mathrm{Ind}_{W_K}^{W_F} \chi)$. Then*

$$\delta^{\mathrm{A}} = \psi_K^{\mathrm{A}} \chi^{\mathrm{A}}|_{F^\times}.$$

In particular, if $\delta = \omega$ then $\psi_K^{\mathrm{A}} \chi^{\mathrm{A}}|_{F^\times} = |\cdot|_v$ and

$$\chi^{\mathrm{A}}|_{\mathcal{O}_F^\times} = \psi_K^{\mathrm{A}}|_{\mathcal{O}_F^\times} \quad \text{and} \quad \chi^s|_{I_K} = \chi^{-1}|_{I_K}.$$

Proof. We have [8, §29.2] (with $d = 2$, $m = 1$)

$$\delta := \det(\mathrm{Ind}_{W_K}^{W_F} \chi) = \psi_K \cdot (\chi \circ \mathrm{ver}_{W_F|W_K}) \tag{2.3.8}$$

where $\mathrm{ver}_{W_F|W_K}$ is the transfer (or Verlagerung) map, fitting in the fundamental commutative diagram from local class field theory:

$$\begin{tikzcd} F^\times \arrow[r] \arrow[d, hook] & W_F^{\mathrm{ab}} \arrow[d, "\mathrm{ver}_{W_F|W_K}"] \\ K^\times \arrow[r] & W_K^{\mathrm{ab}} \end{tikzcd} \tag{2.3.9}$$

(The equality (2.3.8) can also be verified in this case by a direct calculation.) Thus

$$\delta^{\mathrm{A}} = \psi_K^{\mathrm{A}} \chi^{\mathrm{A}}|_{F^\times}. \tag{2.3.10}$$

In particular, we have $\delta = \omega$ if and only if $\delta^{\mathrm{A}} = \psi_K^{\mathrm{A}} \chi^{\mathrm{A}}|_{F^\times} = |\cdot|_v$. When $\delta = \omega$, since ψ_K is quadratic we have $\chi^{\mathrm{A}}|_{\mathcal{O}_F^\times} = (\psi_K^{-1})^{\mathrm{A}}|_{\mathcal{O}_F^\times} = \psi_K^{\mathrm{A}}|_{\mathcal{O}_F^\times}$; and as in Lemma 2.3.2 we have $(\chi^s\chi)^{\mathrm{A}} = \chi^{\mathrm{A}} \circ \mathrm{Nm}_{K|F}$ so restricting to $\mathcal{O}_K^\times$ gives $\chi^s|_{I_K} = \chi^{-1}|_{I_K}$. □

Exceptional supercuspidal representations. The remaining supercuspidal representations are exceptional. A supercuspidal Weil–Deligne representation $\rho\colon W_F \to \mathrm{GL}_2(\mathbb{C})$ is called exceptional (or primitive) if the image of the projective representation $\mathrm{P}\rho$ is isomorphic to A_4 or S_4. Exceptional representations only exist in residual characteristic $p = 2$, so suppose that $p = 2$. We will also use the following characterization. For ρ supercuspidal, let $\mathfrak{I}(\rho)$ be the group of characters $\xi\colon W_F \to \mathbb{C}^\times$ such that $\rho \otimes \xi \simeq \rho$.

Proposition 2.3.11. *Let ρ be a supercuspidal Weil–Deligne representation. Then ρ is exceptional if and only if $\#\mathfrak{I}(\rho) = 1$.*

Proof. See Bushnell–Henniart [8, §41.3]. ☐

This characterization also refines the imprimitive case, as follows. We write D_n for the dihedral group of order $2n$ and C_n for the cyclic group of order n.

Proposition 2.3.12. *Let $\rho = \mathrm{Ind}_{W_K}^{W_F} \chi$ be an imprimitive representation, where $K \supset F$ is a quadratic extension and $\chi\colon W_K \to \mathbb{C}^\times$ a character such that $\chi^s \neq \chi$. Then the following statements hold:*

(a) *$\#\mathfrak{I}(\rho) = 2$ if and only if ρ is induced from a* unique *quadratic extension $K \supset F$.*
(b) *$\#\mathfrak{I}(\rho) = 4$ if and only if ρ has projective image isomorphic to $D_2 \simeq C_2 \times C_2$ and can be induced from three distinct quadratic extensions if and only if χ^s/χ factors through the norm map.*

Proof. See Bushnell–Henniart [8, §41.3, Corollary] and Gérardin [27, Section 2.7]. ☐

Following Propositions 2.3.11 and 2.3.12, we say that a supercuspidal representation ρ is primitive, simply imprimitive, or triply imprimitive according as $\#\mathfrak{I}(\rho) = 1, 2, 4$.

Proposition 2.3.13. *Let ρ be a primitive representation.*

(a) *There exists a cubic extension $L \mid F$ such that $\rho|_{W_L}$ is imprimitive.*
(b) *If $L \mid F$ is cubic Galois, then the representation $\rho|_{W_L}$ is triply imprimitive.*
(c) *If $L \mid F$ is cubic non Galois, let $M \mid F$ be the normal closure of $L \mid F$ and $E \mid F$ the maximal unramified sub-extension of $M \mid F$. Then, the representation $\rho|_{W_L}$ is simply imprimitive, $\rho|_{W_M}$ is triply imprimitive, and $\rho|_{W_E}$ is primitive.*

Proof. See Bushnell-Henniart [8, §42.2, Theorem, p. 258]. ☐

Let $L \supseteq F$ be a (tamely) ramified cubic extension, and let $M \supset L$ be a ramified quadratic extension. (The condition that M is ramified over L is indeed necessary [8, §42.1, Proposition, p. 257, part (1)].) Let χ be a character of W_M such that χ^{A} does not factor through the norm map $\mathrm{Nm}_{M|L}$. Given the data (L, M, χ), by Bushnell–Henniart [8, p. 261] there is a exceptional supercuspidal Weil–Deligne representation $(\rho, 0)$ such that

$$\rho|_{W_L} = \mathrm{Ind}_{W_M}^{W_L} \chi. \tag{2.3.14}$$

Conversely, every exceptional supercuspidal representation is uniquely determined by such a triple (L, M, χ), up to equivalence [11, Lemme 12.1.3].

2.4. **Inertial types.** We now study the restriction to inertia. An inertial Weil–Deligne (or WD-)type is an equivalence class $[\rho, N]$ of Weil–Deligne representations (ρ, N) under the equivalence relation $(\rho, N) \sim (\rho', N')$ if and only if there exists $P \in \mathrm{GL}_2(\mathbb{C})$ such that $\rho'(g) = P\rho(g)P^{-1}$ and $N' = PNP^{-1}$ for all $g \in I_F$. The content is in the restriction to $g \in I_F$; we might think of this as being an equivalence of Weil–Deligne representations over F^{un}. Such an equivalence class is determined by the pair (τ, N) where $\tau = \rho|_{I_F}$ is the (common) restriction to I_F for a WD-type, with the evident notion of equivalence, so this definition agrees with the one given in the introduction. Except for the special (Steinberg) representations we have $N = 0$, so (aside from Section 4.1) we drop N from the notation and write simply τ. The representation ρ is unramified if and only if τ is the trivial inertial type, defined by $\tau(g) = 1$ for all $g \in I_F$.

We record the following classification of all inertial WD-types.

Proposition 2.4.1. *Let $\tau\colon I_F \to \mathrm{GL}_2(\mathbb{C})$ be a nontrivial inertial* WD*-type. Then exactly one of the following holds:*

(i) *τ is the restriction of a principal series, i.e., there exist $\chi_1, \chi_2\colon W_F \to \mathbb{C}^\times$ such that*
$$\tau \simeq \mathrm{PS}(\chi_1, \chi_2)|_{I_F} = \chi_1|_{I_F} \oplus \chi_2|_{I_F};$$
(ii) *$\tau \simeq \mathrm{St}(\chi)|_{I_F}$ is the restriction of a special series for χ a character of W_F;*

(iii) *There exists a character $\chi\colon W_K \to \mathbb{C}^\times$, where $K \supset F$ is the unramified quadratic extension, such that $\chi \neq \chi^s$ and*
$$\tau \simeq (\mathrm{Ind}_{W_K}^{W_F} \chi)|_{I_F} = \chi|_{I_F} \oplus \chi^s|_{I_F};$$
(iv) *There exist a ramified quadratic extension $K \supseteq F$ and a character $\chi\colon W_K \to \mathbb{C}^\times$ such that $\chi|_{I_K} \neq \chi^s|_{I_K}$ and*
$$\tau \simeq \mathrm{Ind}_{I_K}^{I_F}(\chi|_{I_K})$$
is irreducible; or

(v) *τ is the restriction of an exceptional supercuspidal Weil–Deligne representation.*

Proof. For (i)–(iv), see Breuil–Mézard [7, Lemme 2.1.1.2, Théorème 2.1.1.4] and for (v) see Bushnell–Henniart [8, §41 and §42]. ☐

By Proposition 2.4.1, the classification of Weil–Deligne representations in the previous section remains well-defined on inertial types, and so we may accordingly say $[\rho, N]$ is principal series, special, or (nonexceptional or exceptional) supercuspidal.

2.5. **Notation.** We conclude this section with the notation we will use throughout, in one place for convenience.

- The trivial (2-dimensional) type is denoted τ_{triv}.
- We write $\varepsilon_d\colon I_p \to \mathbb{C}^\times$ for the character associated to the (ramified) quadratic extension $\mathbb{Q}_p(\sqrt{d})$ of discriminant $d \in \mathbb{Z}_p$ (more precisely, $d \in \mathbb{Z}_p/\mathbb{Z}_p^{\times 2}$).
- We write $\tau_{\mathrm{St},p}$ to denote the special (Steinberg) type (2.2.2) (not including the nilpotent monodromy operator N in the notation); in all other cases, $N = 0$.
- To identify the nonexceptional supercuspidal types, we use the notation

$$\tau_{\mathrm{sc},p}(d, f, r)_j := \big(\mathrm{Ind}_{W_{\mathbb{Q}_p(\sqrt{d})}}^{W_{\mathbb{Q}_p}} \chi_{(d,f,r)}\big)|_{I_p} \qquad (2.5.1)$$

and for the principal series types we use

$$\tau_{\mathrm{ps},p}(1,f,r) := \chi_{(1,f,r)}|_{I_p} \oplus \chi_{(1,f,r)}^{-1}|_{I_p} \tag{2.5.2}$$

where:

- d is the discriminant of $K := \mathbb{Q}_p(\sqrt{d})$, with $[K:\mathbb{Q}_p] \leq 2$;
- for $p \neq 2$, let $u \in \mathbb{Z}_p^\times \smallsetminus \mathbb{Z}_p^{\times 2}$ be a nonsquare, so $\mathbb{Q}_p(\sqrt{u})$ is the unramified quadratic extension of $\mathbb{Q}_p$ (having discriminant u);
- $\chi_{(d,f,r)} \colon W_K \to \mathbb{C}^\times$ is a character, where:
  * f is the conductor exponent of the character χ (as a power of the maximal ideal in the ring of integers of K);
  * r is the order of the character χ on the inertia subgroup $I_K \subset W_K$; and
  * j is an additional label (only needed for $p = 3$, see Table 3).

- For $p = 2$, two exceptional (octahedral) representations $\tau_{\mathrm{ex},2,i}$ for $i = 1, 2$ are explicitly given (see Section 7.2).

3. Background on elliptic curves

In this section we organize some facts about elliptic curves and provide a few preliminary results on their inertial types. Throughout this section, let E be an elliptic curve over the p-adic field F, and let N_E be the conductor of E.

3.1. **Inertial types.** We begin by defining a Weil–Deligne representation (ρ_E, N) attached to E: for complete details, we refer to Rohrlich [34, §4 and §13–15]. We start with the representation $\rho_{E,\ell} \colon \mathrm{Gal}(F^{\mathrm{al}} \,|\, F) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$ defined by the action of $\mathrm{Gal}(F^{\mathrm{al}} \,|\, F)$ on the étale cohomology group $H^1_{\mathrm{et}}(E \times_F F^{\mathrm{al}}, \mathbb{Q}_\ell) \simeq \mathbb{Z}_\ell^2$ for some prime $\ell \neq p$. We may also work *dually* with the ℓ-adic Tate module, via the isomorphism

$$T_\ell(A) := \varprojlim_n E[\ell^n] \simeq H^1_{\mathrm{et}}(E \times_F F^{\mathrm{al}}, \mathbb{Q}_\ell)^\vee. \tag{3.1.1}$$

The determinant of this representation is the cyclotomic character; so in the principal series case we have $\chi_2 = \chi_1^{-1} \cdot |\cdot|_F$ and in the supercuspidal case, Lemma 2.3.7 applies.

Next, we consider two cases.

- If E has potentially good reduction, then $\rho_{E,\ell}(I_F)$ has finite order. We take $N = 0$ and ρ_E is obtained by extension of scalars of the restriction $\rho_{E,\ell}|_{W_F}$ via an embedding $\iota \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$; the $\mathbb{C}$-equivalence class is well-defined, independent of choices. (See also Remark 2.1.1.)
- Otherwise, E has potentially multiplicative reduction, and so $\rho_{E,\ell}(I_F)$ is infinite. Then E obtains split multiplicative reduction over an at most quadratic extension K. Let χ be the at most quadratic character of W_F attached to K. Then we take $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\rho_E = \mathrm{St}(\chi)$ the Steinberg representation attached to χ.

In either case, we define the inertial WD-type τ_E of E to be the equivalence class $\tau_E = [\rho_E, N]$ as defined in Section 2.4. Finally, we note that the conductors of $\rho_{E,p}$ and τ_E are both equal to N_E, the conductor of E (see e.g. Rohrlich [34, §18] or Darmon–Diamond–Taylor [17, Remark 2.14]).

Example 3.1.2. If E already has good reduction over F, then τ_E is trivial.

Example 3.1.3. For a ramified quadratic extension $\mathbb{Q}_p(\sqrt{d}) \supseteq \mathbb{Q}_p$ of discriminant d, let E_d be the quadratic twist of E over $\mathbb{Q}_p$ by d. Then $\tau_{E_d} \simeq \tau_E \otimes \varepsilon_d$, and twisting by a quadratic character modifies the Galois action only by a scalar. Therefore E and E_d have the same reduction type, and the nilpotent operator N remains unchanged under twist: the monodromy operator N is determined by the unipotent part of the inertia action, which is unaffected by a scalar twist.

The following summarizes the well-known possibilites for N_E in the case $F = \mathbb{Q}_p$.

Lemma 3.1.4. *Let E be an elliptic curve over $\mathbb{Q}_p$. Then*

$$0 \leq \operatorname{ord}_p(N_E) \leq \begin{cases} 2, & \text{if } p \geq 5; \\ 5, & \text{if } p = 3; \\ 8, & \text{if } p = 2. \end{cases}$$

Moreover, if E has additive reduction then $\operatorname{ord}_p(N_E) \geq 2$.

Proof. See e.g. Silverman [37, Theorem IV.10.4]. □

Remark 3.1.5. Elliptic curves defined over *ramified* extensions of $\mathbb{Q}_2$ or $\mathbb{Q}_3$ may have conductors whose valuations are higher than those given by Lemma 3.1.4.

3.2. **Potentially good reduction.** In this section, suppose that E over F has potentially good reduction, and let $\tau = \tau_E$ be its inertial type (with $N = 0$). In this section, we set up some preparatory facts needed in the sequel.

Let $m \in \mathbb{Z}_{\geq 3}$ be coprime to p, and let $L := F^{\mathrm{un}}(E[m])$. The extension L is independent of m (see Serre-Tate [36, §2, Corollary 3]) and it has two other equivalent descriptions:

- L is the minimal extension of F^{un} where E achieves good reduction; and
- L is the fixed field of $\ker \tau$.

We call L the inertial field of E. Write $\Phi := \mathrm{Gal}(L \mid F^{\mathrm{un}})$ and define the semistability defect of E to be $e = e_E := \#\Phi$. The field L is the compositum of F^{un} by a minimal totally ramified extension of F of degree e where E obtains good reduction (though this extension need not be Galois over F), so this definition agrees with the one in the introduction.

The semistability defect can be computed as the ramification degree of $\mathbb{Q}_p(E[m])$, realized using division polynomials.

The following describes the possibilities for Φ.

Lemma 3.2.1. *Exactly one of the following possibilities hold.*

(i) Φ *is cyclic of order* $2, 3, 4, 6$;
(ii) $p = 3$ *and* $\Phi \simeq \mathbb{Z}/3 \rtimes \mathbb{Z}/4$ *is of order* 12;
(iii) $p = 2$ *and* $\Phi \simeq Q_8$ *is isomorphic to a quaternion group of order* 8; *or*
(iv) $p = 2$ *and* $\Phi \simeq \mathrm{SL}_2(\mathbb{F}_3)$ *is of order* 24.

Proof. See Kraus [28, pp. 354–357]. □

Corollary 3.2.2. *Let $p \geq 5$ and suppose E has $p^2 \mid N_E$. Then the semistability defect e is the smallest integer $e \in \{2, 3, 4, 6\}$ such that E obtains good reduction over $\mathbb{Q}_p(\sqrt[e]{p})$.*

Proof. We are in case (i), so Φ is cyclic and the inertial field is tame so given by $\mathbb{Q}_p^{\mathrm{un}}(\sqrt[e]{p})$; but good reduction is invariant under unramified extensions, so it suffices to check over $\mathbb{Q}_p(\sqrt[e]{p})$. □

The above lemma crucially restricts the possibilities in the ramified case, as follows.

Corollary 3.2.3. *Suppose $\tau_E \simeq \mathrm{Ind}_{I_K}^{I_F}(\chi|_{I_K})$ where $K \supset F$ is a ramified quadratic extension and $\chi\colon W_K \to \mathbb{C}^\times$ is a character that does not factor through the norm. Then either $p = 2$ or ($p = 3$ and $e = 12$).*

Proof. If $p \geq 5$, then by Lemma 3.2.1, Φ is cyclic, and so τ_E is reducible. But by Proposition 2.4.1(iv), τ_E is irreducible, a contradiction. If $p = 3$, then we must have case (ii) in Lemma 3.2.1. □

Now suppose that $F = \mathbb{Q}_p$. The next lemma already determines τ in the first nontrivial case.

Lemma 3.2.4. *Suppose E over $F = \mathbb{Q}_p$ has semistability defect $e_E = 2$. Then τ_E is principal series, and the following statements hold.*

(a) *If $p \geq 3$, then $N_E = p^2$ and $\tau_E \simeq \tau_{\mathrm{triv}} \otimes \varepsilon_p$.*
(b) *If $p = 2$, then $N_E = 2^4, 2^6$, and*

$$\tau_E \simeq \begin{cases} \tau_{\mathrm{triv}} \otimes \varepsilon_{-4}, & \text{if } N_E = 2^4; \\ \tau_{\mathrm{triv}} \otimes \varepsilon_{\pm 8}, & \text{if } N_E = 2^6. \end{cases}$$

Proof. The hypothesis that $e = e_E = 2$ means that there exists a ramified quadratic extension $\mathbb{Q}_p(\sqrt{d}) \supseteq \mathbb{Q}_p$ such that the quadratic twist E_d (as in Example 3.1.3) has good reduction (see e.g. Freitas–Kraus [26, Lemmas 3–4]) and therefore the inertial type τ_{E_d} is trivial. Thus

$$\tau \simeq \tau_{E_d} \otimes \varepsilon_d \simeq \tau_{\mathrm{triv}} \otimes \varepsilon_d \simeq \mathrm{PS}(\chi_d, \chi_d)|_{I_p}$$

is principal series. We have $d = p$ if $p \geq 3$; if $p = 2$, we have $d = -4, \pm 8$. The claim on the conductor follows from $\mathrm{ord}_p(N_E) = \mathrm{condexp}(\tau) = 2\,\mathrm{cond}(\chi_d)$ by (2.2.1). □

At the other extreme, we conclude this short section with a preliminary step in classifying exceptional supercuspidal types arising from elliptic curves over $\mathbb{Q}_2$; these will be given explicitly in Section 7.

Lemma 3.2.5. *Suppose $F = \mathbb{Q}_2$ and E has potentially good reduction. Then τ_E is exceptional supercuspidal if and only if $e = 24$.*

Proof. Suppose τ is exceptional. We look at the group structure of the image of the projective representation obtained by postcomposing with $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. By Bushnell–Henniart [8, Section 42.3], we must have semistability defect $e \geq 12$, so in fact $e = 24$ by Lemma 3.2.1.

Conversely, suppose $e = 24$, and let $\rho_{E,3}\colon W_2 \to \mathrm{GL}_2(\mathbb{Q}_3)$ and $\overline{\rho}_3\colon W_2 \to \mathrm{GL}_2(\mathbb{F}_3)$ respectively be the 3-adic and mod 3 Galois representations associated to E, restricted to the Weil–Deligne group W_2. By Dokchitser–Dokchitser [22, Lemma 1], there is an unramified twist of $\rho_{E,3}$ factoring through the Galois group of $K := \mathbb{Q}_2(E[3])$ (over $\mathbb{Q}_2$), so the images of the projective representations $\mathrm{P}\rho_{E,3}\colon W_2 \to \mathrm{PGL}_2(\mathbb{Q}_3)$ and $\mathrm{P}\overline{\rho}_3\colon W_2 \to \mathrm{PGL}_2(\mathbb{F}_3)$ are isomorphic as abstract groups. By hypothesis and Lemma 3.2.1, τ is irreducible with image isomorphic to $\Phi \simeq \mathrm{SL}_2(\mathbb{F}_3)$, so $\overline{\rho}_3(W_2) = \mathrm{GL}_2(\mathbb{F}_3)$ and $\overline{\rho}_3$ is surjective [22, Table 1]. Thus $\mathrm{P}\rho_{E,3}$ has image isomorphic to $\mathrm{PGL}_2(\mathbb{F}_3) \simeq S_4$, so $\rho_{E,3}$ (extended to $\mathrm{GL}_2(\mathbb{C})$) and hence τ_E is exceptional supercuspidal. □

4. Inertial types: uniform cases

We now embark on an explicit and complete description of the inertial types arising from elliptic curves over $\mathbb{Q}_p$. In this section, we treat two cases where the answer is close to uniform in p: potentially multiplicative reduction and $p \geq 5$. Throughout, we use the notation collected in Section 2.5.

4.1. **Potentially multiplicative reduction and special types.** We begin with a general result on inertial types for elliptic curves with potentially multiplicative reduction. These are the only types with a nonzero nilpotent operator as in Section 3.1.

Proposition 4.1.1. *Let E be an elliptic curve over $\mathbb{Q}_p$ with potentially multiplicative reduction, conductor N_E, and inertial type τ_E. Then the following statements hold.*

(a) *If E has multiplicative reduction (over $\mathbb{Q}_p$), then $N_E = p$ and $\tau_E \simeq \tau_{\mathrm{St},p}$ is special.*
(b) *Suppose E has additive (but potentially multiplicative) reduction. Then $p^2 \mid N_E$, and τ_E is special. Moreover:*
  (i) *If $p \geq 3$, then $N_E = p^2$ and $\tau_E \simeq \tau_{\mathrm{St},p} \otimes \varepsilon_p$.*
  (ii) *If $p = 2$, then $N_E = p^4$ or $N_E = p^6$, and*

$$\tau_E \simeq \begin{cases} \tau_{\mathrm{St},2} \otimes \varepsilon_{-4}, & \text{if } N_E = p^4; \\ \tau_{\mathrm{St},2} \otimes \varepsilon_{\pm 8} \text{ or } \tau_{\mathrm{St},2} \otimes \varepsilon_{\pm 8}, & \text{if } N_E = p^6. \end{cases}$$

Proof. We recall from Section 3.1 that $\rho_E = \mathrm{St}(\chi)$ and $\tau = \tau_E = \mathrm{St}(\chi)|_{I_p}$ for some quadratic character χ of W_p.

In part (a), we have $N_E = \mathrm{cond}(\tau) = p$ and by the conductor formula (2.2.3) it follows that χ is unramified; in this case,

$$\tau = (\chi \otimes \mathrm{St}(1))|_{I_p} = \chi|_{I_p} \otimes \mathrm{St}(1)|_{I_p} = \mathrm{St}(1)|_{I_p} = \tau_{\mathrm{St},p}. \tag{4.1.2}$$

We turn to part (b). We have $p^2 \mid N_E$, and formula (2.2.3) gives $N_E = \mathrm{cond}(\tau) = p^{2m}$, so χ is ramified with $\mathrm{cond}(\chi) = p^m$. If $p \geq 3$, then any quadratic character $\chi\colon W_p \to \mathbb{C}^\times$ has conductor p and satisfies $\chi|_{I_p} = \varepsilon_p$. Thus $N_E = p^2$, and $\tau \simeq \tau_{\mathrm{St},p} \otimes \varepsilon_p$, proving (i). Otherwise, we have $p = 2$, and the possibilites are $\chi|_{I_p} = \varepsilon_{-4}, \varepsilon_{\pm 8}$ of conductors $2^2, 2^3$, proving (ii). ☐

4.2. **Inertial types for $p \geq 5$.** The preceding section treated all cases of potentially multiplicative reduction, so for the rest of this paper we turn to the case of potentially good reduction. In particular, $N = 0$. Here we treat the case $p \geq 5$.

Let $e \in \mathbb{Z}_{\geq 3}$ and $\zeta_e := \exp(2\pi i/e)$. We define the following characters:

(1) If $e \mid (p-1)$, let $\chi_{(1,1,e)}\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be of conductor $p\mathbb{Z}_p$ defined on the units (inertia) by $\chi_{(1,1,e)}(u_1) = \zeta_e$ where $u_1 \in \mathbb{Z}_p^\times$ generates $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$.
(2) If $e \mid (p+1)$, let $K := \mathbb{Q}_p(\sqrt{u})$ where u is a quadratic nonresidue modulo p, and define $\chi_{(u,1,e)}\colon K^\times \to \mathbb{C}^\times$ of conductor $p\mathcal{O}_K$ defined on the units (inertia) by $\chi_{(u,1,e)}(u_2) = \zeta_e$ where $u_2 \in \mathcal{O}_K^\times$ generates $(\mathcal{O}_K/p\mathcal{O}_K)^\times$.

Following the notation in Section 2.5, we define the types $\tau_{\mathrm{ps},p}(1,1,e)$ and $\tau_{\mathrm{sc},p}(u,1,e)$ using the characters $\chi_{(1,1,e)}$ and $\chi_{(u,1,e)}$, respectively.

Proposition 4.2.1. *Let $p \geq 5$. Let E be an elliptic curve over $\mathbb{Q}_p$ with additive potentially good reduction, semistability defect $e \geq 3$, and inertial type τ. Then the following statements hold.*

(a) *If* $e \mid (p-1)$*, then* $\tau \simeq \tau_{\mathrm{ps},p}(1,1,e)$ *is principal series.*
(b) *If* $e \mid (p+1)$*, then* $\tau \simeq \tau_{\mathrm{sc},p}(u,1,e)$ *is supercuspidal.*

Proof. Lemma 3.1.4 implies that τ has conductor p^2, and Lemma 3.2.1 shows that the image of τ is cyclic of order $e = 3,4,6$. From the classification in Proposition 2.4.1, τ is reducible with finite image, hence it is either principal series or nonexceptional supercuspidal induced from the unramified quadratic extension $K = \mathbb{Q}_p(\sqrt{u})$ of $\mathbb{Q}_p$.

Suppose that τ is principal series. Then, $\tau = \chi|_{I_p} \oplus \chi^{-1}|_{I_p}$, where χ is a character of W_p of conductor p and order e. To ease notation we write χ also for χ^{A}. Thus, $\chi|_{I_p}$ factors through $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times \simeq \mathbb{F}_p^\times$ a cyclic group of order $p-1$, so $e \mid (p-1)$. Let $u_1 \in \mathbb{Z}_p^\times$ be as above. We have $\chi(u_1) = \zeta_e^c = \exp(2\pi i c/e)$ with $\gcd(c,e) = 1$. Since $e = 3,4,6$, we must have $c \equiv \pm 1 \pmod{e}$, so $\chi|_{I_p} = \chi_{(1,1,e)}^{\pm 1}$ and either choice gives $\tau \simeq \tau_{\mathrm{ps},p}(1,1,e)$.

To finish, suppose τ is supercuspidal, obtained by induction of a character χ of W_K of order e on I_K. Since τ has conductor $p^2\mathcal{O}_K$, it follows that χ^{A} viewed as a character of $K^\times$ has conductor $p\mathcal{O}_K$. Moreover, by Lemma 2.3.7, we have $\tau = \chi|_{I_K} \oplus \chi^{-1}|_{I_K}$ and $\chi^{\mathrm{A}}|_{\mathbb{Z}_p^\times} = 1$ (as K is unramified over $\mathbb{Q}_p$). Now $\chi|_{I_K}$ factors through $(\mathcal{O}_K/p\mathcal{O}_K)^\times \simeq \mathbb{F}_{p^2}^\times$ therefore $e \mid (p^2-1) = (p+1)(p-1)$. Let $u_2 \in \mathcal{O}_K^\times$ be as above. Then u_2^{p+1} generates $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times \leq (\mathbb{Z}_{p^2}/p\mathbb{Z}_{p^2})^\times$. The condition $\chi|_{\mathbb{Z}_p^\times} = 1$ implies that $e \mid (p+1)$. We thus have $\chi(u_2) = \zeta_e^c$ and similarly to the previous paragraph we must have $\chi|_{I_K} = \chi_{(u,1,e)}^{\pm 1}$ and either choice gives $\tau \simeq \tau_{\mathrm{sc},p}(u,1,e)$. $\square$

5. Inertial types for $p = 3$

In this section, we treat the case $p = 3$; see Breuil–Conrad–Diamond–Taylor [6] for the application of these types to the modularity of elliptic curves.

5.1. **Setup.** Throughout this section, we let $K := \mathbb{Q}_3(\sqrt{d})$ where $d = \pm 1, \pm 3$: when $d = 1$ we have $K = \mathbb{Q}_3$. Let $\mathcal{O}_K$ be the valuation ring of K and $\mathfrak{p}$ its maximal ideal. When $K \neq \mathbb{Q}_3$, let $\chi_d \colon W_3 \to \mathbb{C}^\times$ be the quadratic character associated to K and let $s \in W_3$ be a lift of the nontrivial element of $\mathrm{Gal}(K \mid \mathbb{Q}_3)$. Recall (Section 2.5) that $\varepsilon_{-1} = \chi_{-1}|_{I_3}$ is trivial (the extension $\mathbb{Q}_3(\sqrt{-1}) \supseteq \mathbb{Q}_3$ is unramified) and $\varepsilon_3 = \chi_{\pm 3}|_{I_3}$ is the unique nontrivial quadratic character of I_3. We have $\mathrm{cond}(\varepsilon_3) = 3$. For $d = -3$, let $\xi_6 := (1+\sqrt{-3})/2 \in K$ (to help distinguish it from roots of unity in $\mathbb{C}^\times$).

We begin with basic structural results concerning unit groups. Let $\mathfrak{f} = \mathfrak{p}^f$ with $f \geq 1$, and let $(q) = \mathbb{Z} \cap \mathfrak{f}$ where $q = 3^m$ with $m > 0$. More precisely, we have $m = f$ when K is unramified and $m = \lfloor f/2 \rfloor$ when K is ramified. We have seen throughout the above that the norm map plays an important role. Write

$$\mathrm{Nm} = \mathrm{Nm}_{K|\mathbb{Q}_3} \colon (\mathcal{O}_K/\mathfrak{f})^\times \to (\mathbb{Z}_3/q\mathbb{Z}_3)^\times$$

for the norm map (the identity when $K = \mathbb{Q}_3$). We also use the following abbreviation for its image after embedding it back $(\mathbb{Z}_3/q\mathbb{Z}_3)^\times \hookrightarrow (\mathcal{O}_K/\mathfrak{f})^\times$:

$$U_{\mathfrak{f}} := \begin{cases} \{1\}, & \text{if } K = \mathbb{Q}_3; \\ \mathrm{Nm}((\mathcal{O}_K/\mathfrak{f})^\times), & \text{otherwise.} \end{cases} \tag{5.1.1}$$

Lemma 5.1.2. *Table 2 gives the structure and explicit generators for the groups* $U_{\mathfrak{f}}$ *and* $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$*, respectively.*

K	d	$\mathfrak{f}$	f	$U_{\mathfrak{f}}$	$(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$
$\mathbb{Q}_3$	1	$(3)^f$	≥ 1	$-$	$\langle -4\rangle \simeq \mathbb{Z}/(2\cdot 3^{f-1})$
$\mathbb{Q}_3(\sqrt{-1})$	-1	$(3)^f$	≥ 1	$\langle -4\rangle \simeq \mathbb{Z}/(2\cdot 3^{f-1})$	$\langle \sqrt{-1}+2\rangle \simeq \mathbb{Z}/(4\cdot 3^{f-1})$
$\mathbb{Q}_3(\sqrt{3})$	3	$\mathfrak{p}^f$	≥ 1	$\langle 4\rangle \simeq \mathbb{Z}/(3^{\lfloor (f-1)/2\rfloor})$	$\langle \sqrt{3}-1\rangle \simeq \mathbb{Z}/(2\cdot 3^{\lfloor f/2\rfloor})$
$\mathbb{Q}_3(\sqrt{-3})$	-3	$\mathfrak{p}^f$	$1,2,3$	$\langle 4\rangle \simeq \mathbb{Z}/(3^{\lfloor (f-1)/2\rfloor})$	$\langle \xi_6-3\rangle \simeq \mathbb{Z}/(2\cdot 3^{\lfloor f/2\rfloor})$
			≥ 4	$\langle 4\rangle \simeq \mathbb{Z}/(3^{\lfloor (f-1)/2\rfloor})$	$\langle -\xi_6\rangle \times \langle \xi_6-3\rangle \simeq \mathbb{Z}/3\times \mathbb{Z}/(2\cdot 3^{\lfloor (f-2)/2\rfloor})$

TABLE 2. Group structure of $U_{\mathfrak{f}}$ and $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$

For our main result, we only need the description in the last column of this table for finitely many values of f, which can be computed directly: the code is online [18]. Still, we thought it was interesting to show the uniformity in the description.

Proof. The case $K=\mathbb{Q}_3$ is elementary, so we suppose $[K:\mathbb{Q}_3]=2$.

We first compute the column $U_{\mathfrak{f}}$. The group $(\mathbb{Z}_3/q\mathbb{Z}_3)^\times$ is cyclic; and the image of the norm at least contains the subgroup of squares of index 2 generated by 4, with the nontrivial coset represented by -1. By Hensel's lemma (solving the equation $\mathrm{Nm}(x+y\sqrt{d})=x^2-dy^2=c$), the image of the norm contains -1 if and only if it does so modulo 3 if and only if K is unramified over $\mathbb{Q}_3$. The order of the cyclic quotient group is determined by computing $\mathfrak{p}^f\cap\mathbb{Z}_p$ in the three cases.

We now treat the final column. First, we computed [18] elements of small height whose images generate $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$ for the small conductors and the structure of this group in Magma. To finish, we follow ideas of Cohen [13, Proof of Theorem 4.2.10, Corollary 4.2.11].

We explain this in the hardest case $K=\mathbb{Q}_3(\sqrt{-3})$; the other two rows follow similarly (and a bit more easily). We prove that $4, -\xi_6, \xi_6-3$ generate $(\mathcal{O}_K/\mathfrak{f})^\times$ for all $f\geq 4$. We first compute this for $f=4$ and we also calculate that

(5.1.3) $\qquad 4^3, (\xi_6-3)^6$ generate the group $(1+\mathfrak{p}^4)/(1+p\mathfrak{p}^4)\simeq \mathcal{O}_K/(3)\simeq \mathbb{Z}/3\times\mathbb{Z}/3.$

Next, for $f>\mathrm{ord}_{\mathfrak{p}}(p)/(p-1)=1$, the p-adic logarithm gives an isomorphism $\log\colon 1+\mathfrak{p}^f \xrightarrow{\sim} \mathfrak{p}^f\simeq\mathbb{Z}_p^2$. So for $f=4$ we have by Nakayama's lemma topological generators for $1+\mathfrak{p}^f$ hence generators for $(1+\mathfrak{p}^4)/(1+\mathfrak{p}^f)$ as well. We finish from the exact sequence

$$1\to(1+\mathfrak{p}^4)/(1+\mathfrak{p}^f)\to(\mathcal{O}_K/\mathfrak{p}^f)^\times\to(\mathcal{O}_K/\mathfrak{p}^4)^\times\to 1;$$

our elements give both the subgroup and the quotient group, so they generate $(\mathcal{O}_K/\mathfrak{p}^f)^\times$.

To finish, we compute the group structure of the quotient for $f\geq 4$. Of course $-\xi_6$ has order 3, so what remains is to compute the order of ξ_6-3; but it generates the further quotient by $-\xi_6$ which is cyclic, so its order is

$$[(\mathcal{O}_K/\mathfrak{f})^\times : U_{\mathfrak{f}}\langle -\xi_6\rangle] = 2\cdot p^{f-2-\lfloor (f-1)/2\rfloor} = 2\cdot p^{\lfloor (f-2)/2\rfloor} \tag{5.1.4}$$

as claimed. ∎

Remark 5.1.5. From the last column of Table 2, we see that for $K=\mathbb{Q}_3$, $\mathbb{Q}_3(\sqrt{-1})$ or $\mathbb{Q}_3(\sqrt{3})$, the quotient $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$ is a cyclic group. So the canonical projection

$$(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}\to(\mathcal{O}_K/\mathfrak{f}')^\times/U_{\mathfrak{f}'},$$

when $\mathfrak{f}' \mid \mathfrak{f}$, behave as expected. For $K = \mathbb{Q}_3(\sqrt{-3})$, however, this is not quite the case. Indeed, in that case, the quotient $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$ is no longer cyclic for $\mathfrak{f}$ large enough. In particular, when $\mathfrak{f} = \mathfrak{p}^4$ and $\mathfrak{f}' = \mathfrak{p}^3$, the projection

$$\pi : (\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f} \to (\mathcal{O}_K/\mathfrak{f}')^\times/U_{\mathfrak{f}'}$$

sends $-\xi_6 \mapsto (\xi_6 - 3)^4 \pmod{\mathfrak{f}'}$ and $\xi_6 - 3 \mapsto \xi_6 - 3 \pmod{\mathfrak{f}'}$. So $\pi(-\xi_6) \neq 1$, as one might be tempted to think from the description of the groups involved.

Corollary 5.1.6. *Let $\chi\colon W_3 \to \mathbb{C}^\times$ be a nontrivial quadratic character. Then $\chi^{\mathrm{A}}(4) = 1$, and $\chi = \varepsilon_{-1}$ if and only if $\chi^{\mathrm{A}}(-1) = 1$. In particular, χ is ramified if and only if $\chi^{\mathrm{A}}(-1) = -1$.*

Proof. By the first row of Table 2, for χ quadratic we must have $\chi^{\mathrm{A}}(4) = 1$ and so by local class field theory we have $\chi^{\mathrm{A}}(-1) = 1$ if and only if χ is unramified. ☐

Corollary 5.1.7. *Let $\chi\colon W_K \to \mathbb{C}^\times$ be a character with K ramified and $U_\mathfrak{f} \leq \ker \chi^{\mathrm{A}}$. Then $\operatorname{condexp}(\chi)$ is even.*

Proof. One way to prove the corollary is just to observe that this follows from properties of the floor function, applied to the last column of Table 2.

Alternatively, here is a more satisfying argument which holds for odd p and $K \supseteq \mathbb{Q}_p$ quadratic and ramified. Consider the natural projection $\pi\colon (\mathcal{O}_K/\mathfrak{p}^f)^\times \to (\mathcal{O}_K/\mathfrak{p}^{f-1})^\times/U_{\mathfrak{p}^{f-1}}$. Suppose $f = \operatorname{condexp}(\chi)$ is odd. We claim that this map has kernel $U_{\mathfrak{p}^f}$; the result follows, as then χ factors through $(\mathcal{O}_K/\mathfrak{p}^{f-1})^\times/U_{\mathfrak{p}^{f-1}}$, which contradicts the definition of conductor exponent. By compatibility we have $\ker \pi \geq U_{\mathfrak{p}^f}$, so we show the other inclusion. To that end, suppose that $\pi(u) = \operatorname{Nm}(u') \in U_{\mathfrak{p}^{f-1}}$. Lifting u' to $(\mathcal{O}_K/\mathfrak{p}^f)^\times$ arbitrarily, we get $u/\operatorname{Nm}(u') \equiv 1 \pmod{\mathfrak{p}^{f-1}}$ so $u/\operatorname{Nm}(u') \equiv 1 + \sqrt{d}^{f-1} u_1 \pmod{\mathfrak{p}^f}$ for some $u_1 \in \mathcal{O}_K/\mathfrak{p} = \mathbb{Z}/p$. But f is odd, so $\sqrt{d}^{f-1} = d^{(f-1)/2} \in \mathbb{Z}_p$; but then

$$\operatorname{Nm}(1 + \sqrt{d}^{f-1} u_1/2) = (1 + d^{(f-1)/2} u_1/2)^2 \equiv 1 + d^{(f-1)/2} u_1 \pmod{\mathfrak{p}^f}$$

as desired. ☐

5.2. **Result.** Returning to the classification in Proposition 2.4.1, and recalling that we have already treated the case of potentially multiplicative reduction in Proposition 4.1.1 and that the exceptional case only happens for $p = 2$, in this section we classify the inertial types for elliptic curves over $\mathbb{Q}_3$ with potentially good reduction, giving principal series ($d = 1$) or (nonexceptional) supercuspidal ($d = -1, \pm 3$) and associated to a character as in (2.5.2) and (2.5.1), respectively.

Recall our attempt at uniform and informative notation: we write our character of W_K as $\chi_{(d,f,r)}$, where K has discriminant d, with r the order when restricted to I_K and f the conductor exponent. We work explicitly with such characters, taking values in $\mathbb{C}^\times$ in terms of roots of unity denoted $\zeta_k := \exp(2\pi i/k)$ for $k \geq 1$. For the purposes of the restriction of inertia, we only care about $\chi_{(d,f,r)}|_{I_K}$, so to spell out the type we will describe $\chi^{\mathrm{A}}_{(d,f,r)}|_{\mathcal{O}_K^\times}$. By definition of conductor exponent, this restriction then factors through $(\mathcal{O}_K/\mathfrak{f})^\times$ where $\mathfrak{f} = \mathfrak{p}^f$, so can be given on generators as in the previous section.

- In the principal series case, $U_\mathfrak{f}$ is trivial by definition.
- In the supercuspidal case, we are in the situation of Lemma 2.3.7, which reads

$$\chi^{\mathrm{A}}_{(d,f,r)}|_{\mathbb{Z}_3^\times} = \varepsilon_d. \tag{5.2.1}$$

K	d	$\mathfrak{f}$	f	r	values of χ on generators	τ	$\operatorname{condexp}(\tau)$
$\mathbb{Q}_3$	1	3^2	2	3	ζ_3	$\tau_{\mathrm{ps},3}(1,2,3)$	4
$\mathbb{Q}_3(\sqrt{-1})$	-1	(3)	1	4	ζ_4	$\tau_{\mathrm{sc},3}(-1,1,4)$	2
		$(3)^2$	2	3	ζ_3	$\tau_{\mathrm{sc},3}(-1,2,3)$	4
$\mathbb{Q}_3(\sqrt{3})$	3	$\mathfrak{p}^2$	2	6	ζ_6	$\tau_{\mathrm{sc},3}(3,2,6)$	3
$\mathbb{Q}_3(\sqrt{-3})$	-3	$\mathfrak{p}^2$	2	6	ζ_6	$\tau_{\mathrm{sc},3}(-3,2,6)$	3
		$\mathfrak{p}^4$	4	6	$\zeta_3^j,\ -\zeta_3^{j-1}$	$\tau_{\mathrm{sc},3}(-3,4,6)_j$ $(j=0,1,2)$	5

TABLE 3. Nonspecial inertial types for $\mathbb{Q}_3$

So by local class field theory, $U_{\mathfrak{f}} \le \ker \chi^{\mathrm{A}}_{(d,f,r)}$.

In either case, we may specify the character by its values on the generators of $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$ given in Table 2. In all cases but the last row of this table, the character is uniquely determined by the data (d,f,r) (up to Galois conjugation), sending the designated generator to ζ_r. For the last row, we will write $\chi_{(-3,4,6),j}$ to be the character which takes the values $\chi(-\xi_6) = \zeta_3^j$ and $\chi(\xi_6 - 3) = -\zeta_3^{j-1}$ for $j = 0,1,2$. We summarize this data in Table 3.

Proposition 5.2.2. *Let E be an elliptic curve over $\mathbb{Q}_3$ of conductor N_E and inertial type τ_E. Suppose that E has additive, potentially good reduction and semistability defect $e \ge 3$. Then τ_E is given by one of the following cases:*

(a) *If $N_E = 3^2$, then $e = 4$ and $\tau_E \simeq \tau_{\mathrm{sc},3}(-1,1,4)$.*
(b) *If $N_E = 3^3$, then $e = 12$ and $\tau_E \simeq \tau_{\mathrm{sc},3}(3,2,6)$ or $\tau_E \simeq \tau_{\mathrm{sc},3}(-3,2,6)$.*
(c) *If $N_E = 3^4$, then $e = 3,6$, and:*
  (i) *If $e = 3$, then $\tau_E \simeq \tau_{\mathrm{ps},3}(1,2,3)$ or $\tau_E \simeq \tau_{\mathrm{sc},3}(-1,2,3)$;*
  (ii) *If $e = 6$, then $\tau_E \simeq \tau_{\mathrm{ps},3}(1,2,3) \otimes \varepsilon_3$ or $\tau_E \simeq \tau_{\mathrm{sc},3}(-1,2,3) \otimes \varepsilon_3$.*
(d) *If $N_E = 3^5$, then $e = 12$ and $\tau_E \simeq \tau_{\mathrm{sc},3}(-3,4,6)_j$ with $j = 0,1,2$.*

Proof. We drop subscripts, writing e.g. $\tau = \tau_E$. We have $N_E = 3^k$ with $2 \le k = \operatorname{condexp}(\tau) \le 5$ by Lemma 3.1.4.

We begin with (a), and suppose $k = 2$. Then we are in the case of tame reduction, and Φ is cyclic of order $e = 4$ by Lemma 3.2.1. We cannot have τ principal series, since then $\tau \simeq \chi|_{I_3} \oplus \chi^{-1}|_{I_3}$ with χ a character of W_3 of conductor exponent 1 and order 4, which does not exist. So τ must be (nonexceptional) supercuspidal. Since $e = 4 \mid (3+1)$, we conclude that $\tau \simeq \tau_{\mathrm{sc},3}(-1,1,4)$ by a similar argument as in the proof of Proposition 4.2.1(b).

Next, we prove part (b) and (d) and suppose $k = 3$ or $k = 5$. Since $k > 1$ is odd, by (2.3.5), τ is obtained by induction of a character χ from a ramified quadratic extension $K \supset \mathbb{Q}_3$ with $d = \pm 3$ and $m = \operatorname{condexp}(\chi) + v_3(d) = \operatorname{condexp}(\chi) + 1$. In particular, we are in case (iv) of Proposition 2.4.1: τ is irreducible and in this case it is necessary and sufficient for $(\chi|_{I_K})^{\mathrm{A}}$ not to factor via the norm. Hence, $e = 12$ by Corollary 3.2.3. This gives three cases:

- Suppose $k = 3$, so $\operatorname{condexp}(\chi) = 2$. By Lemma 5.1.2, we have
$$(\mathcal{O}_K/\mathfrak{p}^2)^\times/U_{\mathfrak{f}} = \langle u \rangle \simeq \mathbb{Z}/6,$$
where $u = \sqrt{3} - 1$ for $d = 3$ and $u = \xi_6 - 3$ for $d = -3$. Since χ is primitive with $\operatorname{condexp}(\chi) = 2$, we must have $\chi(u) = \pm\zeta_3^j$ with $j = 1,2$. Applying Corollary 5.1.6

to $\chi|_{\mathbb{Z}_3^\times} = \varepsilon_d$ we conclude that $\chi(-1) = -1$ and hence $\chi(u) = -\zeta_3^j$ with $j = 1$ or $j = 2$, which are Galois conjugate. Note that χ does not factor via the norm by Corollary 2.3.4(a), so the type indeed occurs. We obtain the same induction from either character, so we may take $\chi|_{I_K} = \chi_{(d,2,6)}|_{I_K}$, giving $\tau \simeq \tau_{\mathrm{sc},3}(d, 2, 6)$.

- Next, suppose that $k = 5$ and $d = 3$. By Lemma 5.1.2, $\chi|_{I_K}$ is a nontrivial character on
$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U_{\mathfrak{f}} = \langle u\rangle \simeq \mathbb{Z}/18.$$
The order of $\chi(u)$ is not a divisor of 6, otherwise χ would be imprimitive (i.e., $\mathrm{condexp}(\chi) \leq 3$); thus $\chi|_{I_K}$ has order 9 or 18 and so $9 \mid e = 12$, a contradiction. Therefore this case does not occur.
- Finally, suppose $k = 5$ and $d = -3$. Now by Lemma 5.1.2 we have a character on
$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U_{\mathfrak{f}} = \langle u_1\rangle \times \langle u_2\rangle \simeq \mathbb{Z}/3 \times \mathbb{Z}/6$$
where $u_1 = -\xi_6$ and $u_2 = \xi_6 - 3$. The primitivity condition here is more subtle. Indeed, if χ is imprimitive then $\chi = \theta \circ \pi$ for some character θ with $\mathrm{condexp}(\theta) \leq 3$, where π is the projection defined in Remark 5.1.5. Since $(\mathcal{O}_K/\mathfrak{p}^3)^\times/U_{\mathfrak{f}} \simeq \mathbb{Z}/6$ we conclude that $\chi(u_1u_2^2) = \theta(u_2^6) = 1$. Thus χ is primitive if and only if $\chi(u_1) \neq \chi(u_2)^{-2}$. Furthermore, Corollary 5.1.6 applied to $\chi|_{\mathbb{Z}_3^\times} = \varepsilon_d$ gives $\chi(-1) = \chi(u_2^3) = -1$ and so $\chi|_{I_K}$ does not factor through the norm by Corollary 2.3.4(a). We conclude that $\chi(u_2) = \zeta_6^i = (-1)^i\zeta_3^{2i}$ with $i = 1, 3, 5$ and the constraints on $\chi(u_1)$ follow from the primitivity condition. More precisely, up to replacing χ by its Galois conjugate (which coincides with χ^{-1} on inertia), there are three possible characters defined by
$$\chi(u_1) = \zeta_3^j \quad \text{and} \quad \chi(u_2) = -\zeta_3^{j-1} \quad \text{for } j = 0, 1, 2.$$
Therefore, we can choose $\chi = \chi_{(-3,4,6)_j}$ for $j = 0, 1, 2$. Thus $\tau \simeq \tau_{\mathrm{sc},3}(-3, 4, 6)_j$.

We conclude by proving (c). Suppose $k = 4$.

- First, suppose that τ is principal series. Then $\tau = \chi|_{I_3} \oplus \chi^{-1}|_{I_3}$, where χ is a character of W_3 with conductor exponent 2. By Lemma 5.1.2, we have that $\chi|_{I_3}$ factors through
$$(\mathbb{Z}_3/3^2\mathbb{Z}_3)^\times \simeq \langle -4\rangle \simeq \mathbb{Z}/6.$$
Since χ is primitive with $\mathrm{condexp}(\chi) = 2$, we have $\chi(4) = \zeta_3^{\pm 1}$. Twisting by ε_3, we can assume that $\chi(-1) = 1$. Thus, there are two possibilities for $\chi|_{I_3}$, which are $\chi_{(1,2,3)}$ and $\chi_{(1,2,3)}^{-1}$. Thus $\tau \simeq \tau_{\mathrm{ps},3}(1, 2, 3)$ (hence $e = 3$) or $\tau \simeq \tau_{\mathrm{ps},3}(1, 2, 3) \otimes \varepsilon_3$ (hence $e = 6$).
- We are left with the case where τ is supercuspidal. We first claim that K is unramified: indeed, if K is ramified, then again from (2.3.5) we have $4 = \mathrm{condexp}(\chi) + \mathrm{condexp}(\psi_K) = \mathrm{condexp}(\chi) + 1$ so $\mathrm{condexp}(\chi) = 3$. But this contradicts Corollary 5.1.7. Thus K is unramified, so $d = -1$, and $\mathrm{condexp}(\chi) = 2$ by (2.3.5). Reading Lemma 5.1.2 one last time, we have $\chi|_{I_K}$ on
$$\left(\mathcal{O}_K/(3^2)\right)^\times/U_{\mathfrak{f}} = \langle u\rangle \simeq \mathbb{Z}/12$$
where $u = \sqrt{-1} + 2$. By Corollary 5.1.6, $\chi(4) = \chi(-1) = 1$. By primitivity, $\chi(u)$ must have order 3, 6, or 12. However, by Lemma 3.2.1, the case $e = 12$ only arises for nonabelian inertia, which is not possible as $K \supset \mathbb{Q}_3$ is unramified. Thus

τ	e	$\operatorname{condexp}(\tau)$	$\mathrm{Nm}_{L_\tau\|K}(\mathcal{O}_{L_\tau}^\times)/U_\mathfrak{f}$	L'	$\mathrm{Gal}(L_\tau\,\|\,\mathbb{Q}_3)$	E
trivial	1	0	–	3.1.0.1	1T1 $\simeq C_1$	11a1
ε_3	2	2	–	3.1.0.1	2T1 $\simeq C_2$	99d2
$\tau_{\mathrm{ps},3}(1,2,3)$	3	4	–	3.3.4.2	3T1 $\simeq C_3$	162b1
$\tau_{\mathrm{sc},3}(-1,2,3)$	3	4	$\langle 3\rangle \simeq \mathbb{Z}/4$	3.3.4.4	3T2 $\simeq S_3$	162d1
$\tau_{\mathrm{sc},3}(-1,1,4)$	4	2	trivial	3.4.3.1	4T3 $\simeq D_4$	36a1
$\tau_{\mathrm{ps},3}(1,2,3)\otimes\varepsilon_3$	6	4	–	3.6.9.11	6T1 $\simeq C_6$	162c2
$\tau_{\mathrm{sc},3}(-1,2,3)\otimes\varepsilon_3$	6	4	$\langle 6\rangle \simeq \mathbb{Z}/2$	3.6.9.12	6T3 $\simeq D_6$	162a2
$\tau_{\mathrm{sc},3}(3,2,6)$	12	3	trivial	3.12.15.1	12T15 $\simeq 2\cdot D_6$	27a1
$\tau_{\mathrm{sc},3}(-3,2,6)$	12	3	trivial	3.12.15.12	12T13 $\simeq 2\cdot D_6$	54a1
$\tau_{\mathrm{sc},3}(-3,4,6)_0$	12	5	$\langle(1,0)\rangle \simeq \mathbb{Z}/3$	3.12.23.122	12T13 $\simeq 2\cdot D_6$	972b1
$\tau_{\mathrm{sc},3}(-3,4,6)_1$	12	5	$\langle(0,2)\rangle \simeq \mathbb{Z}/3$	3.12.23.20	12T13 $\simeq 2\cdot D_6$	243b1
$\tau_{\mathrm{sc},3}(-3,4,6)_2$	12	5	$\langle(1,4)\rangle \simeq \mathbb{Z}/3$	3.12.23.14	12T13 $\simeq 2\cdot D_6$	243a1

TABLE 4. Types, defining fields, and elliptic curves realizing each inertial type over $\mathbb{Q}_3$ in the case of potentially good reduction, where $2\cdot D_6 \simeq C_4 : C_3 \simeq (C_6\times C_2) : C_2$

$\chi(u)=\zeta_6^j$, with $j\in\{1,2,4,5\}$, which give two pairs of conjugate characters. Note that Corollary 2.3.4(b) implies that indeed χ does not factor through the norm.

To finish, we recognize these two as twists. We note that $\delta=\chi\cdot(\varepsilon_3|_{W_K})$ is also a character of W_K with conductor exponent 2 with $\delta(g)$ having order 3 or 6, and moreover

$$\delta(u)=\chi(u)\chi_3(\mathrm{Nm}_{K|\mathbb{Q}_3}(u))=\chi(u)\varepsilon_3(u^2)=\chi(u)=1$$

for all $u\in\mathbb{Z}_3^\times$. Therefore, $\delta|_{\mathcal{O}_K^\times}$ must be one of the previous four characters. Thus, up to twisting by ε_3, we can assume that $\chi(g)=\zeta_6^j$, for $j=2,4$, which is the same as requiring that $\chi(g)=\zeta_3^{\pm1}$ which are now conjugate. Thus we can take $\chi|_{I_K}=\chi_{(-1,2,3)}$, and either $\tau\simeq\tau_{\mathrm{sc},3}(-1,2,3)$ with $e=3$ or $\tau\simeq\tau_{\mathrm{sc},3}(-1,2,3)\otimes\varepsilon_3$ with $e=6$.

The proof is then complete by exhaustion of cases. ☐

5.3. **Explicit realization.** In Table 4, we give an example of a curve realizing each inertial type we computed over $\mathbb{Q}_3$ for the case of potentially good reduction; in particular those described in Proposition 5.2.2 and Table 3. Additional columns are explained below.

To this end, let τ be an inertial type. Let $L'\supseteq\mathbb{Q}_p$ be an extension of minimal degree such that $\mathbb{Q}_p^{\mathrm{un}}L'$ is the extension of $\mathbb{Q}_p^{\mathrm{un}}$ cut out by τ. If $[L':\mathbb{Q}_p]=[\mathbb{Q}_p^{\mathrm{un}}L':\mathbb{Q}_p^{\mathrm{un}}]$, then we call L' a descent of the inertial field of τ. Let L_τ be the normal closure of a descent field of τ.

When τ is a non-exceptional supercuspidal type, there is a quadratic subfield $K\subset L_\tau$ and a character $\chi\colon W_K\to\mathbb{C}^\times$ such that, by class field theory,

$$\ker\chi^A=\mathrm{Nm}_{L_\tau|K}(L_\tau^\times)\subset K^\times \tag{5.3.1}$$

and the restriction to inertia $\chi|_{I_K}$ satisfies

$$\ker(\chi^A|_{(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}})=\mathrm{Nm}_{L_\tau|K}(\mathcal{O}_{L_\tau}^\times)/U_\mathfrak{f}\hookrightarrow(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}. \tag{5.3.2}$$

Proposition 5.3.3. *For $p = 3$ and each inertial type τ arising from an elliptic curve with additive, potentially good reduction, there is a unique descent L' of the inertial field. Moreover, either $L' = L_\tau$ is Galois or the compositum $L_\tau = \mathbb{Q}_9 L'$ is Galois over $\mathbb{Q}_3$, where $\mathbb{Q}_9$ is the quadratic unramified extension of $\mathbb{Q}_3$.*

Proof. There are 12 such types; we can exclude the trivial case leaving 11. Accompanying code is available online [18].

By search, we find a list of 11 elliptic curves E over $\mathbb{Q}_3$ and a list of 11 nonisomorphic totally ramified extensions L' such that for each elliptic curve E there is a unique field L' of minimal degree such that $E_{L'}$ has good reduction. (Indeed, most curves obtain good reduction over exactly one listed field; some obtain good reduction over two or more fields but only one with minimal degree.) Since the fields L' are totally ramified and of minimal degree they form a list of uniquely determined (up to conjugation in $\overline{\mathbb{Q}}_p$) descent fields.

We verify the second statement, that the fields are either Galois over $\mathbb{Q}_3$ or become so after taking the compositum with the quadratic unramified extension; we then verify that the extensions are pairwise nonisomorphic as extensions of $\mathbb{Q}_9$ hence as extensions of $\mathbb{Q}_3^{\mathrm{un}}$.

The fields L' are listed in Table 4 by LMFDB label. Moreover, for each nonabelian Galois closure $L \supseteq \mathbb{Q}_3$ of a L' in our list, we computed the image of $\mathrm{Nm}_{L|K}(\mathcal{O}_L^\times)/U_\mathfrak{f}$ inside $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$ where the latter is given by the group structure and generators in Table 2. (For some of the fields there are more than one quadratic subfield $K \subset L$; the choice of K in the 4th column of Table 4 is the field K used to define the corresponding type in the first column, which turns out to be the correct choice as explained below.)

Given that we have a distinct list of fields and a list of types with the same cardinality, the result follows. $\square$

To finish the proof of Table 4 we need to match the descent fields with the types. Since the curve `11a1` has good reduction at 3 and its quadratic twist by 3 is the curve `99d2`, the first two rows follow immediately. The principal series types correspond by construction to abelian extensions $L = L' \supseteq \mathbb{Q}_3$. Thus $\tau_{\mathrm{ps},3}(1,2,3)$ corresponds to the unique cyclic extension in the list with ramification $e = 3$ and $\tau_{\mathrm{ps},3}(1,2,3) \otimes \varepsilon_3$ to the unique cyclic extension with $e = 6$.

We are then left with the supercuspidal types. The fields L' are totally ramified and their Galois closure is obtained by the compositum with at most an unramified quadratic extension, so the computed norm groups $\mathrm{Nm}_{L|K}(\mathcal{O}_L^\times)/U_\mathfrak{f}$ uniquely identify it as an extension of K by local class field theory. To select the quadratic subextension $K \subseteq L$ and the conductor $\mathfrak{f}$ for the calculation of the norm groups, we used the following two facts:

(1) The conductor of a type τ with corresponding L_τ matches the conductor of the curve that corresponds to L_τ.
(2) The description of the types in the list with a fixed conductor gives the candidates quadratic subextensions $K \subset L_\tau$; for example, Proposition 5.2.2(d) implies that fields corresponding to curves of conductor 3^5 are obtained from characters of $K = \mathbb{Q}_3(\sqrt{-3})$ (even though $\mathbb{Q}_3(\sqrt{3})$ and $\mathbb{Q}_3(\sqrt{-1})$ are also contained in those fields).

Comparing the output of this calculation to the definition of the characters in Table 3 completes the proof of all the rows except for with $\mathrm{condexp}(\tau_E) = 3$. This is because the fields corresponding to `27a1` and `54a1` both contain the two quadratic fields $\mathbb{Q}_3(\sqrt{\pm 3})$ and there is one type of conductor 3^3 induced from each of these quadratic fields, namely

$\tau_{\mathrm{sc},3}(\pm 3,2,6)$. So to finish, we observe that for $\tau_{\mathrm{sc},3}(\pm 3,2,6)$ the induced characters have order 6 and $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f} \simeq \mathbb{Z}/6$, hence the norm group must be trivial as per the table. Further, this norm group can only occur if we are taking norms towards a quadratic field from which the type can be induced from. Thus if the trivial norm group occurs also for the other quadratic field then the type must be triply imprimitive by Proposition 2.3.12. But this is not the case because its Galois group is isomorphic to 24.8 of order 24, a group which also goes by the names

$$2 \cdot D_6 \simeq (C_6 \times C_2) : C_2 \simeq C_3 : D_4;$$

however, this group has center C_2 and the quotient modulo center has order 12, hence the projectivization of its image cannot be isomorphic to $D_2 \simeq C_2 \times C_2$.

Having realized types explicitly this way, we obtain the following corollary.

Corollary 5.3.4. *Let E be an elliptic curve over $\mathbb{Q}_3$ with potentially good reduction. Then there is a unique field L' in Table 4 of minimal degree such that E obtains good reduction over L', and τ_E is given by the type that corresponds to this L'.*

Code implementing Corollary 5.3.4 is available online [18].

6. Nonexceptional inertial types at $p = 2$

In this section, we begin our consideration of inertial types for the case $p = 2$. We treat all inertial types but for the exceptional supercuspidal types, leaving the latter for Section 7. The outline of the argument is the same as in the case $p = 3$, just with more technical complications.

6.1. Setup and statement of result.

In this section, we let $K = \mathbb{Q}_2(\sqrt{d})$ be one of the eight at most quadratic extensions of $\mathbb{Q}_2$, so $d = 1, -4, 5, \pm 8, -20, \pm 40$. The unique nontrivial unramified extension has $d = 5$; the remaining nontrivial extensions have conductor exponent 2 or 3. As before, when $K \neq \mathbb{Q}_2$ let $s \in W_2$ be a lift of the nontrivial element of $\mathrm{Gal}(K \,|\, \mathbb{Q}_2)$.

As in the case $p = 3$, we will need to know the quadratic character ε_d on inertia explicitly, as follows.

- We have $\varepsilon_{-4} = \varepsilon_{-20}$, with $\mathrm{condexp}(\varepsilon_{-4}) = 2$ and its restriction $\varepsilon_{-4}^{\mathrm{A}}|_{\mathbb{Z}_2^\times}$ factors through $(\mathbb{Z}_2/2^2\mathbb{Z}_2)^\times$ and
$$\varepsilon_{-4}(-1) = -1.$$
- We have $\varepsilon_8 = \varepsilon_{40}$, with $\mathrm{condexp}(\varepsilon_8) = 3$ and similarly ε_8 is defined on $(\mathbb{Z}_2/2^3\mathbb{Z}_2)^\times = \langle -1 \rangle \times \langle 5 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ by
$$\varepsilon_8(-1) = 1, \quad \varepsilon_8(5) = -1.$$
- We have $\varepsilon_{-8} = \varepsilon_8\varepsilon_{-1} = \varepsilon_{-40}$, with $\mathrm{condexp}(\varepsilon_8) = 3$ and
$$\varepsilon_{-8}(-1) = -1, \quad \varepsilon_{-8}(5) = -1.$$

As in Section 5, we begin by setting up the structures of finite quotients of unit groups. We adopt the same notation: $\mathfrak{f} = \mathfrak{p}^f$ with $f \geq 1$ and $(q) = \mathbb{Z}_2 \cap \mathfrak{f}$ where $q = 2^k$ for some $k > 0$. Also, $\mathrm{Nm} = \mathrm{Nm}_{K|\mathbb{Q}_2} \colon (\mathcal{O}_K/\mathfrak{f})^\times \to (\mathbb{Z}_2/q\mathbb{Z}_2)^\times$ the norm map, and finally

$$U_\mathfrak{f} := \begin{cases} \{1\}, & \text{if } K = \mathbb{Q}_2; \\ \mathrm{Nm}((\mathcal{O}_K/\mathfrak{f})^\times), & \text{otherwise.} \end{cases}$$

K	d	$\mathfrak{f}$	f	$U_{\mathfrak{f}}$
$\mathbb{Q}_2$	1	$(2)^f$	≥ 1	–
$\mathbb{Q}_2(\sqrt{5})$	5	(2)	1	trivial
		$(2)^2$	2	$\langle -1\rangle \simeq \mathbb{Z}/2$
		$(2)^f$	≥ 3	$\langle -1\rangle \times \langle 3\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2^{f-2}$
$\mathbb{Q}_2(\sqrt{m})$, $m=-1,-5$	$-4,-20$	$\mathfrak{p}^f$	$1,2,3,4$	trivial
		$\mathfrak{p}^f$	≥ 5	$\langle -3\rangle \simeq \mathbb{Z}/2^{\lfloor (f-3)/2\rfloor}$
$\mathbb{Q}_2(\sqrt{m})$, $m=2,10$	$8,40$	$\mathfrak{p}^f$	$1,2$	trivial
		$\mathfrak{p}^f$	$3,4,5,6$	$\langle -1\rangle \simeq \mathbb{Z}/2$
		$\mathfrak{p}^f$	≥ 7	$\langle -1\rangle \times \langle -9\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor (f-5)/2\rfloor}$
$\mathbb{Q}_2(\sqrt{m})$, $m=-2,-10$	$-8,-40$	$\mathfrak{p}^f$	$1,2$	trivial
		$\mathfrak{p}^f$	$3,4,5,6$	$\langle 3\rangle \simeq \mathbb{Z}/2$
		$\mathfrak{p}^f$	≥ 7	$\langle 3\rangle \simeq \mathbb{Z}/2^{\lfloor (f-3)/2\rfloor}$

TABLE 5. Group structure of $U_{\mathfrak{f}}$

K	d	$\mathfrak{f}$	f	$(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$
$\mathbb{Q}_2$	1	(2)	1	trivial
		$(2)^2$	2	$\langle -1\rangle \simeq \mathbb{Z}/2$
		$(2)^f$	≥ 3	$\langle -1\rangle \times \langle 5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2^{f-2}$
$\mathbb{Q}_2(\sqrt{5})$	5	$(2)^f$	$1,2$	$\langle (-1+\sqrt{5})/2\rangle \simeq \mathbb{Z}/(3\cdot 2^{f-1})$
			≥ 3	$\langle \sqrt{5}\rangle \times \langle (-1+\sqrt{5})/2\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/(3\cdot 2^{f-2})$
$\mathbb{Q}_2(\sqrt{m})$, $m=-1,-5$	$-4,-20$	$\mathfrak{p}$	1	trivial
		$\mathfrak{p}^2$	2	$\langle \sqrt{m}\rangle \simeq \mathbb{Z}/2$
		$\mathfrak{p}^f$	≥ 3	$\langle \sqrt{m}\rangle \times \langle 2\sqrt{m}-1\rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/2^{\lfloor (f-2)/2\rfloor}$
$\mathbb{Q}_2(\sqrt{m})$, $m=\pm 2,\pm 10$	$\pm 8,\pm 40$	$\mathfrak{p}^f$	$1,2,3,4$	$\langle \sqrt{m}-1\rangle \simeq \mathbb{Z}/2^{\lfloor f/2\rfloor}$
			≥ 5	$\langle -3\rangle \times \langle \sqrt{m}-1\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor f/2\rfloor}$

TABLE 6. Group structure of $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$

Lemma 6.1.1. *Tables 5, 6, and 7 give the structure and generators for the groups $U_{\mathfrak{f}}$, $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$, and $(\mathbb{Z}_2/q\mathbb{Z}_2)^\times/U_{\mathfrak{f}}$, respectively.*

Again, for the main result we only need these tables for finitely many values of f.

Proof. The argument for Tables 5 and 6 are entirely similar to the proof of Lemma 5.1.2; see the code.

Completing Table 7 can be read from the previous two; we illustrate in one example. Suppose $m = 2, 10$. Since K is ramified, then $q = 2^k$ where $k := \lceil f/2\rceil$. From Table 6 we have:

- If $f = 1, 2$ then $k = 1$, and both $U_{\mathfrak{f}}$ and $(\mathbb{Z}/2^k)^\times$ are trivial.
- If $f = 3, 4$ then $k = 2$, and $U_{\mathfrak{f}} = \langle -1\rangle$ and $(\mathbb{Z}/2^k)^\times = \langle -1\rangle$.

K	d	$\mathfrak{f}$	f	$(\mathbb{Z}_2/q\mathbb{Z}_2)^\times/U_\mathfrak{f}$
$\mathbb{Q}_2$	1	(2)	1	trivial
		$(2)^2$	2	$\langle -1\rangle \simeq \mathbb{Z}/2$
		$(2)^f$	≥ 3	$\langle -1\rangle \times \langle 5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2^{f-2}$
$\mathbb{Q}_2(\sqrt{5})$	5	$(2)^f$	≥ 1	trivial
$\mathbb{Q}_2(\sqrt{m})$, $m=-1,-5$	$-4,-20$	$\mathfrak{p}^f$	$1,2$	trivial
			≥ 3	$\langle -1\rangle \simeq \mathbb{Z}/2$
$\mathbb{Q}_2(\sqrt{m})$, $m=2,10$	$8,40$	$\mathfrak{p}^f$	$1,2,3,4$	trivial
			≥ 5	$\langle -3\rangle \simeq \mathbb{Z}/2$
$\mathbb{Q}_2(\sqrt{m})$, $m=-2,-10$	$-8,-40$	$\mathfrak{p}^f$	$1,2,3,4$	trivial
			≥ 5	$\langle -1\rangle \simeq \mathbb{Z}/2$

TABLE 7. Group structure of $(\mathbb{Z}_2/q\mathbb{Z}_2)^\times/U_\mathfrak{f}$

- If $f = 5, 6$ then $k = 3$, we have $U_\mathfrak{f} = \langle -1\rangle$, $(\mathbb{Z}/2^k)^\times = \langle -1\rangle \times \langle 5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.
- Finally, for $f \geq 7$, we have

$$\left\lfloor \frac{f-5}{2} \right\rfloor = \left\lceil \frac{f}{2} \right\rceil - 2 = k - 2.$$

Therefore, $U_\mathfrak{f}$ contains exactly half the elements of $(\mathbb{Z}_2/q\mathbb{Z}_2)^\times$.

Therefore, the quotient $(\mathbb{Z}_2/q\mathbb{Z}_2)^\times/U_\mathfrak{f}$ is trivial for $1 \leq f \leq 4$ and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $f \geq 5$, proving the 4th row of Table 7. The remaining rows follow similarly. $\square$

Remark 6.1.2. Similarly to Remark 5.1.5, the canonical projection maps between quotients behave as expected, except for the field $K = \mathbb{Q}_2(\sqrt{5})$. In that case, the projection map

$$\pi : (\mathcal{O}_K/\mathfrak{p}^3)^\times/U_\mathfrak{f} \to (\mathcal{O}_K/\mathfrak{p}^2)^\times/U_\mathfrak{f}$$

is given by

$$\pi(\sqrt{5}) = \left(\frac{-1+\sqrt{5}}{2}\right)^3 \pmod{\mathfrak{p}^2}, \qquad \pi\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{-1+\sqrt{5}}{2} \pmod{\mathfrak{p}^2}.$$

So $\pi(\sqrt{5}) \neq 1$ contrary to what one might think at first glance.

Corollary 6.1.3. *Let χ be a character of $\mathbb{Q}_2(\sqrt{d})^\times$ with conductor $\mathfrak{f} = \mathfrak{p}^f$.*

(a) *For $d = 5$, we have*

$$\chi|_{\mathbb{Z}_2^\times} = \varepsilon_5 \iff \chi|_{U_\mathfrak{f}} = 1 \iff \chi(3) = \chi(-1) = 1.$$

(b) *For $d = -4, -20$ and $f \geq 3$, we have*

$$\chi|_{\mathbb{Z}_2^\times} = \varepsilon_{-4} \iff \chi|_{U_\mathfrak{f}} = 1 \text{ and } \chi(-1) = -1 \iff \chi(3) = \chi(-1) = -1.$$

(c) *For $d = 8, 40$ and $f \geq 5$, we have*

$$\chi|_{\mathbb{Z}_2^\times} = \varepsilon_8 \iff \chi|_{U_\mathfrak{f}} = 1 \text{ and } \chi(-3) = -1 \iff \chi(3) = -1 \text{ and } \chi(-1) = 1.$$

K	d	$\mathfrak{f}$	f	r	values of χ on generators	τ	condexp(τ)
$\mathbb{Q}_2$	1	$(2)^4$	4	4	$1,\ i$	$\tau_{\mathrm{ps},2}(1,4,4)$	8
$\mathbb{Q}_2(\sqrt{5})$	5	(2)	1	3	ζ_3	$\tau_{\mathrm{sc},2}(5,1,3)$	2
		$(2)^4$	4	4	$1,\ i$	$\tau_{\mathrm{sc},2}(5,4,4)$	8
$\mathbb{Q}_2(\sqrt{-1})$	-4	$\mathfrak{p}^3$	3	4	i	$\tau_{\mathrm{sc},2}(-4,3,4)$	5
		$\mathfrak{p}^6$	6	4	$i,\ i$	$\tau_{\mathrm{sc},2}(-4,6,4)$	8
$\mathbb{Q}_2(\sqrt{-5})$	-20	$\mathfrak{p}^3$	3	4	i	$\tau_{\mathrm{sc},2}(-20,3,4)$	5

TABLE 8. Nonexceptional, nonspecial inertial types over $\mathbb{Q}_2$

N_E	e_E	τ_E
2^2	3	$\tau_{\mathrm{sc},2}(5,1,3)$
2^4	6	$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-4}$
2^5	8	$\tau_{\mathrm{sc},2}(-4,3,4),\ \tau_{\mathrm{sc},2}(-20,3,4)$
2^6	6	$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{8},\ \tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-8}$
	8	$\tau_{\mathrm{sc},2}(-4,3,4)\otimes\varepsilon_{8},\ \tau_{\mathrm{sc},2}(-20,3,4)\otimes\varepsilon_{8}$
2^8	4	$\tau_{\mathrm{ps},2}(1,4,4),\ \tau_{\mathrm{ps},2}(1,4,4)\otimes\varepsilon_{-4},\ \tau_{\mathrm{sc},2}(5,4,4),\ \tau_{\mathrm{sc},2}(5,4,4)\otimes\varepsilon_{-4}$
	8	$\tau_{\mathrm{sc},2}(-4,6,4),\ \tau_{\mathrm{sc},2}(-4,6,4)\otimes\varepsilon_{8}$

TABLE 9. Inertial types over $\mathbb{Q}_2$ with semistability defect $e\neq 2,24$

(d) *For $d=-8,-40$ and $f\geq 5$, we have*

$$\chi|_{\mathbb{Z}_2^\times}=\varepsilon_{-8} \iff \chi|_{U_\mathfrak{f}}=1 \text{ and } \chi(-1)=-1 \iff \chi(3)=1 \text{ and } \chi(-1)=-1.$$

Proof. This follows from Lemma 6.1.1. ☐

We now focus on the cases of principal series and nonexceptional supercuspidal, where we attached a character $\chi_{(d,f,r)}$ of W_K where K has discriminant d, whose conductor exponent is f and whose order on I_K is r. As in the case $p=3$, to define $\chi_{(d,f,r)}|_{I_K}$ satisfying $\chi_{(d,f,r)}|_{\mathbb{Z}_2^\times}=\varepsilon_d$, it is enough to give the values of $\chi_{(d,f,r)}$ on generators of $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$.

In preparation for our final result, we list the relevant characters in this way in Table 8.

The main result of this section is then as follows.

Theorem 6.1.4. *Let E be an elliptic curve over $\mathbb{Q}_2$ with additive, potentially good reduction, conductor N_E, semistability defect e_E, and inertial type τ_E. Suppose that $e_E\neq 2,24$. Then τ_E is given by one of the cases in Table 9.*

In particular, τ_E is nonexceptional supercuspidal in all cases except when

$$e_E=4 \text{ and } N_E=2^8 \text{ and } \tau_E\simeq\tau_{\mathrm{ps},2}(1,4,4),\tau_{\mathrm{ps},2}(1,4,4)\otimes\varepsilon_{-4}$$

in which case τ_E is principal series.

The proof of Theorem 6.1.4 will be given in Section 6.6 after treating various cases in the next few sections.

6.2. **Principal series.** Throughout the remaining sections, let E be an elliptic curve over $\mathbb{Q}_2$ with additive, potentially good reduction, semistability defect $e_E \neq 2, 24$, conductor N_E, and inertial type τ_E.

We begin with the relatively easy case of principal series.

Proposition 6.2.1. *If τ_E is principal series, then $N_E = 2^8$, $e = 4$, and $\tau_E \simeq \tau_{\mathrm{ps},2}(1,4,4)$ or $\tau_E \simeq \tau_{\mathrm{ps},2}(1,4,4) \otimes \varepsilon_{-4}$.*

Proof. By Lemma 3.1.4 and (2.2.1), we have $\operatorname{condexp}(\tau_E) = 2k$ with $1 \leq k \leq 4$. Thus $\tau_E = \chi|_{I_2} \oplus \chi^{-1}|_{I_2}$, where $\chi|_{I_2}$ factors through $(\mathbb{Z}_2/2^k\mathbb{Z}_2)^\times$. From Lemma 6.1.1 and Table 6 we see that if $k \leq 3$, then $\chi|_{I_2}$ is at most quadratic, so $e \leq 2$, a contradiction. Therefore, $k = 4$ and $\chi|_{I_2}$ factors through

$$(\mathbb{Z}_2/2^4\mathbb{Z}_2)^\times = \langle -1 \rangle \times \langle 5 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4.$$

The primitivity of χ forces $\chi(5) = \pm i$. Twisting by ε_{-4} we can assume $\chi(-1) = 1$. Thus $\chi|_{I_2} = \chi_{(1,4,4)}$ or $\chi_{(1,4,4)}^{-1}$. We conclude that $\tau \simeq \tau_{\mathrm{ps},2}(1,4,4)$ or $\tau \simeq \tau_{\mathrm{ps},2}(1,4,4) \otimes \varepsilon_{-4}$. $\square$

6.3. **Quadratic inductions, conductor 8.** Having dealt with the reducible case, we consider in the remaining subsections inertial types induced from a quadratic extension $K \supset \mathbb{Q}_2$. In this section, we rule out the possibility that K has conductor exponent 3.

Proposition 6.3.1. *Suppose that τ_E is irreducible and induced from a quadratic extension $K \supset \mathbb{Q}_2$. Then K is either unramified or it has conductor exponent 2.*

Proof. Assume for purposes of contradiction that τ_E is induced from a quadratic extension $K \supset \mathbb{Q}_2$ of conductor exponent 3, i.e., $d = \pm 8, \pm 40$, equivalently, $K = \mathbb{Q}_2(\sqrt{m})$ with $m = \pm 2, \pm 10$. By Proposition 2.4.1(iv), we have $\tau \simeq \operatorname{Ind}_{I_K}^{I_2}(\chi|_{I_K})$ where $\chi|_{I_K} \neq \chi^s|_{I_K}$. By Lemma 3.2.1, the order of τ_E is $e_E = 8$ as τ is irreducible and $e \neq 24$; in particular, $\Phi \simeq Q_8$ is quaternion of order 8.

Let $f := \operatorname{condexp}(\chi)$. The conductor exponent formula $\operatorname{condexp}(\tau) = 3 + f$ and Lemma 3.1.4 imply $f \leq 5$. From Lemma 6.1.1 we have that $\chi|_{I_K}$ factors through

$$(\mathcal{O}_K/\mathfrak{p}^f)^\times/U_{\mathfrak{f}} = \begin{cases} \langle u \rangle \simeq \{1\}, & \text{if } f = 1; \\ \langle u \rangle \simeq \mathbb{Z}/2, & \text{if } f = 2, 3; \\ \langle u \rangle \simeq \mathbb{Z}/4, & \text{if } f = 4; \\ \langle u \rangle \times \langle -3 \rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/2, & \text{if } f = 5; \end{cases} \tag{6.3.2}$$

where $u = \sqrt{m} - 1$.

We claim that χ^s/χ is quadratic. By the above, we have χ^s/χ nontrivial. From (6.3.2) and Lemma 2.3.6 it follows that χ^s/χ is at most quadratic on inertia. On the other hand, for the uniformizer $\sqrt{m} \in K$, we have

$$(\chi^s)^{\mathrm{A}}(\sqrt{m}) = \chi^{\mathrm{A}}(-\sqrt{m}) = \varepsilon_d^{\mathrm{A}}(-1)\chi^{\mathrm{A}}(\sqrt{m})$$

whence

$$(\chi^s/\chi)^{\mathrm{A}}(\sqrt{m}) = \varepsilon_d^{\mathrm{A}}(-1)$$

and so χ^s/χ is at most quadratic—hence quadratic.

By Corollary 2.3.4(c), we conclude that χ^s/χ factors through the norm map. By Proposition 2.3.12, we conclude that ρ_E is triply imprimitive, so the projective image of ρ_E is $D_2 = C_2 \times C_2$. On the other hand, by Dokchitser–Dokchitser [22, Lemma 1], there is a twist ρ

of ρ_E which factors through $\mathbb{Q}_2(E[3])$, so $\mathrm{P}\rho_E \simeq \mathrm{P}\rho \simeq \mathrm{P}\overline{\rho}_{E,3}$. But then $\overline{\rho}_{E,3}(W_2) \leq \mathrm{GL}_2(\mathbb{F}_3)$ is the 2-Sylow subgroup by [22, Table 1] which has order 16, hence $\mathrm{P}\overline{\rho}_{E,3}(W_2)$ has order at least 8, and so $\mathrm{P}\overline{\rho}_{E,3}(W_2) \not\simeq C_2 \times C_2$, giving a contradiction. ☐

6.4. **Quadratic unramified inductions.** We now consider the case of inertial types induced from the unramified quadratic extension $K = \mathbb{Q}(\sqrt{5})$.

Proposition 6.4.1. *Suppose τ_E is nonexceptional supercuspidal, obtained by inducing a character χ of W_K where $K = \mathbb{Q}_2(\sqrt{5})$. Then τ_E is given by one of the following cases:*

(a) *If $N_E = 2^2$, then $e_E = 3$ and $\tau_E \simeq \tau_{\mathrm{sc},2}(5,1,3)$.*
(b) *If $N_E = 2^4$, then $e_E = 6$ and $\tau_E \simeq \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$.*
(c) *If $N_E = 2^6$, then $e_E = 6$ and either $\tau_E \simeq \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_8$ or $\tau_E \simeq \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-8}$.*
(d) *If $N_E = 2^8$, then $e_E = 4$ and either $\tau_E \simeq \tau_{\mathrm{sc},2}(5,4,4)$ or $\tau_E \simeq \tau_{\mathrm{sc},2}(5,4,4) \otimes \varepsilon_{-4}$.*

Proof. Since K is unramified over $\mathbb{Q}_2$, we are in case (iii) of Proposition 2.4.1(iii). We conclude that τ has cyclic image of order $e_E = 3, 4, 6$ by Lemma 3.2.1 (having excluded $e_E = 2$ and $e_E = 24$ at the start). The conductor exponent of τ is equal to $2k$ by (2.3.5), where $k := \mathrm{condexp}(\chi) \leq 4$ by Lemma 3.1.4.

From Corollary 6.1.3 we have $\chi|_{\mathbb{Z}_2^\times} = \varepsilon_5$ is trivial. Moreover, since the order of $\chi|_{I_2}$ is also $e > 2$ it follows from Corollary 2.3.4(b) that all the potential candidates for $\chi|_{I_2}$ arising below do not factor via the norm map (as required for irreducibility). Let $\nu := (-1+\sqrt{5})/2$.

- Suppose $k = 1$; then $N_E = 2^2$, and the reduction is tame, hence $e_E = 3$. From Lemma 6.1.1 and Table 6, the character $\chi|_{I_2}$ factors through
$$(\mathcal{O}_K/\mathfrak{p})^\times/U_\mathfrak{f} = \langle \nu \rangle \simeq \mathbb{Z}/3.$$
So $\chi(\nu) = \zeta_3^{\pm 1}$ and indeed χ does not factor through the norm. Thus there are two conjugated possibilities for $\chi|_{I_2}$, and we can take $\chi|_{I_2} = \chi_{(5,1,3)}$, which gives $\tau \simeq \tau_{\mathrm{sc},2}(5,1,3)$. This handles (a).
- Suppose $k = 2$. From Lemma 6.1.1 and Table 6, $\chi|_{I_2}$ factors through
$$(\mathcal{O}_K/\mathfrak{p}^2)^\times/U_\mathfrak{f} = \langle \nu \rangle \simeq \mathbb{Z}/6.$$
From this, we see that χ is primitive if and only if $\chi(\nu) = \zeta_6^{\pm 1}$ and again, χ does not factor via the norm. Thus there are two conjugate choices for $\chi|_{I_{\mathbb{Q}_2}}$, giving rise to the same type τ', showing that there is a unique type of conductor 2^4. But, twisting an elliptic curve with inertial type $\tau_{\mathrm{sc},2}(5,1,3)$ by -1, gives an elliptic curve of conductor 2^4 and inertial type $\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$. Since τ' is the unique inertial type of conductor 2^4, we must have $\tau' = \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$. This proves (b).
- Suppose $k = 3$. From Lemma 6.1.1 and Table 6, $\chi|_{I_2}$ factors through the quotient
$$(\mathcal{O}_K/\mathfrak{p}^3)^\times/U_\mathfrak{f} = \langle \sqrt{5} \rangle \times \langle \nu \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/6.$$
The primitivity condition here is more subtle; indeed, if $\chi = \theta \circ \pi$ for some character θ with $\mathrm{condexp}(\theta) \leq 2$ where π is the projection defined in Remark 6.1.2 then, since $(\mathcal{O}_K/\mathfrak{p}^2)^\times/U_\mathfrak{f} \simeq \mathbb{Z}/6$ we conclude that $\chi(\sqrt{5} \cdot \nu^3) = \theta(\nu^6) = 1$. Thus χ is primitive if and only if $\chi(\sqrt{5}) \neq \chi(\nu)^{-3}$. Moreover, since $\mathbb{Z}/4$ is not a subgroup of the above quotient, we have $e = 3$ or $e = 6$ and so $\chi(\nu) = \zeta_6^j$ with $1 \leq j \leq 5$ and $3 \neq j$. Thereofre, if $j = 1, 5$ then $\chi(\sqrt{5}) = 1$ from the primitivity condition and if $j = 2, 4$

then $\chi(\sqrt{5}) = -1$. This gives four possibilities for $\chi|_{I_2}$ yielding two pairs of conjugate characters, and hence two possible inertial types of conductor 2^6. But, twisting an elliptic curve with inertial type $\tau_{\mathrm{sc},2}(5,1,3)$ by 2 and -2 gives an elliptic curve of conductor 2^6 and inertial type $\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_8$ and $\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-8}$, respectively. So, $\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_8$ and $\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-8}$ must be the two inertial types of conductor 2^6, completing the proof of (c).

- Finally, suppose $k = 4$. From Lemma 6.1.1 and Table 6, $\chi|_{I_2}$ factors through the quotient
$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U_{\mathfrak{f}} = \langle\sqrt{5}\rangle \times \langle\nu\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/12.$$
Since χ is primitive, $\chi(\nu)$ has order 4 or 12. In the latter case, the image of τ has size $e = 12$, a contradiction. So $\chi(\nu) = \pm i$ and χ does not factor via the norm. Since there are no further constraints we can have $\chi(\sqrt{5}) = \pm 1$. This gives four possible characters. Similarly, one can show that $\delta = \chi \cdot \varepsilon_{-4}|_K$ has conductor $\mathfrak{p}^4$, satisfies $\delta|_{\mathbb{Z}_2^\times} = 1$ and that it does not factor through the norm. Hence the possibilities for $\chi|_{I_2}$ are
$$\chi_{(5,4,4)},\ \chi^s_{(5,4,4)},\ \chi_{(5,4,4)} \cdot \varepsilon_{-4},\ \text{or } (\chi_{(5,4,4)} \cdot \varepsilon_{-4})^s,$$
therefore $\tau_E \simeq \tau_{\mathrm{sc},2}(5,4,4)$ or $\tau_E \simeq \tau_{\mathrm{sc},2}(5,4,4) \otimes \varepsilon_{-4}$, as desired. (Note that the twisted types $\tau_{\mathrm{sc},2}(5,4,4)\otimes\varepsilon_{\pm 8}$ also have conductor 2^8 but they do not appear above due to the relations $\chi^s_{(5,4,4)} = \chi_{(5,4,4)}\varepsilon_{-8}$ and $(\chi_{(5,4,4)}\varepsilon_{-4})^s = \chi_{(5,4,4)}\varepsilon_8$.)

This exhausts the possible cases and completes the proof. □

6.5. **Quadratic inductions, conductor 4.** We conclude with the case of conductor $4 = 2^2$.

Proposition 6.5.1. *Suppose τ_E is nonexceptional supercuspidal type, obtained by inducing a character χ of W_K where $K \supseteq \mathbb{Q}_2$ has conductor exponent 2. Then $e_E = 8$ and τ_E is given by one of the following cases:*

(a) *If $N_E = 2^5$, then $\tau_E \simeq \tau_{\mathrm{sc},2}(-4,3,4)$ or $\tau_E \simeq \tau_{\mathrm{sc},2}(-20,3,4)$.*
(b) *If $N_E = 2^6$, then $\tau_E \simeq \tau_{\mathrm{sc},2}(-4,3,4)\otimes\varepsilon_8$ or $\tau_E \simeq \tau_{\mathrm{sc},2}(-20,3,4)\otimes\varepsilon_8$.*
(c) *If $N_E = 2^8$, then $\tau_E \simeq \tau_{\mathrm{sc},2}(-4,6,4)$ or $\tau_E \simeq \tau_{\mathrm{sc},2}(-4,6,4)\otimes\varepsilon_8$.*

Proof. By assumption $e \neq 24$, hence by Lemma 3.2.1 and Proposition 2.4.1 we must have $e_E = 8$.

The quadratic extensions $K \supset \mathbb{Q}_2$ of conductor 2^2 are $K = \mathbb{Q}_2(\sqrt{m})$ with $m = -1, -5$ corresponding to $d = -4, -20$. In both cases, we must have $\chi|_{\mathbb{Z}_2^\times} = \varepsilon_{-4}$, which implies that $\chi(-1) = \chi(3) = -1$ by Corollary 6.1.3. In particular, all the candidates for χ we will find below do not factor via the norm map by Corollary 2.3.4(a). By the conductor exponent formula (2.3.5) and Lemma 3.1.4, we have $k := \operatorname{condexp}(\chi) \leq 6$. Since all characters with conductor exponent ≤ 2 have $\chi(-1) = 1$, we must have $3 \leq k \leq 6$. Furthermore, from Table 6 we also see there are no primitive characters for $k = 5$ for both values of m, hence $k = 3, 4, 6$.

- Suppose $d = -4$ and $k = 3$. By Lemma 6.1.1 and Table 6, $\chi|_{I_K}$ factors through
$$(\mathcal{O}_K/\mathfrak{p}^3)^\times/U_{\mathfrak{f}} = \langle\sqrt{m}\rangle \simeq \mathbb{Z}/4.$$
Since χ is primitive, we must have $\chi(\sqrt{m}) = \pm i$. Therefore, there are two possible conjugate choices. Taking $\chi|_{I_K} = \chi_{(-4,3,4)}$, we obtain $\tau \simeq \tau_{\mathrm{sc},2}(-4,3,4)$. This proves (a) for $m = -1$.

- Suppose $d = -4$ and $k = 4$. By Lemma 6.1.1 and Table 6, $\chi|_{I_K}$ factors through
$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U_\mathfrak{f} = \langle\sqrt{m}\rangle \times \langle 2\sqrt{m}-1\rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/2.$$
Since χ is primitive, we have $\chi(2\sqrt{m}-1) = -1$. Furthermore, since $\chi(-1) = \chi(\sqrt{m})^2 = -1$, we must have $\chi(\sqrt{m}) = \pm i$. Therefore, there are two conjugate choices, and a unique type of conductor 2^6. But, twisting an elliptic curve with inertial type $\tau_{\mathrm{sc},2}(-4,3,4)$ by 2 gives an elliptic curve with conductor 2^6 and inertial type $\tau_{\mathrm{sc},2}(-4,3,4) \otimes \varepsilon_8$. By the uniqueness of the type at conductor 2^6, we must have $\tau \simeq \tau_{\mathrm{sc},2}(-4,3,4) \otimes \varepsilon_8$, proving the first type in (b).
- Suppose $d = -4$ and $k = 6$. By Lemma 6.1.1 and Table 6, $\chi|_{I_K}$ factors through
$$(\mathcal{O}_K/\mathfrak{p}^6)^\times/U_\mathfrak{f} = \langle\sqrt{m}\rangle \times \langle 2\sqrt{m}-1\rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/4.$$
Since χ is primitive, we must have $\chi(2\sqrt{m}-1) = \pm i$. Analogously to the case $k = 4$, we also have $\chi(\sqrt{m}) = \pm i$. This gives rise to four characters. Note that $\delta = \chi \cdot \varepsilon_8|_K$ also has conductor $\mathfrak{p}^6$, and satisfies $\delta|_{\mathbb{Z}_2^\times} = \varepsilon_{-4}$ and does not factor through the norm. We conclude that the four possibilities for $\chi|_{I_K}$ are
$$\chi_{(-4,6,4)}, \quad \chi^s_{(-4,6,4)}, \quad \chi_{(-4,6,4)} \cdot \varepsilon_8|_K, \quad (\chi_{(-4,6,4)} \cdot \varepsilon_8|_K)^s,$$
yielding $\tau = \tau_{\mathrm{sc},2}(-4,6,4)$ or $\tau_{\mathrm{sc},2}(-4,6,4) \otimes \varepsilon_8$. This proves (c) for $d = -4$.
- Suppose $d = -20$ and $k = 3, 4$. Then the same argument as for $m = -1$ applies, using again Lemma 6.1.1 and Table 6 for the group structures. This completes the proof of (a) and (b) for $d = -20$.
- Finally, suppose $d = -20$ and $k = 6$. By Lemma 6.1.1 and Table 6, $\chi|_{I_K}$ factors through
$$(\mathcal{O}_K/\mathfrak{p}^6)^\times/U_\mathfrak{f} = \langle\sqrt{m}\rangle \times \langle 2\sqrt{m}-1\rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/4.$$
Since χ is primitive we have $\chi(2\sqrt{m}-1) = \pm i$ as for $m = -1$. We also have
$$\chi(\sqrt{m})^2 = \chi(-5) = \chi(3) = -1 \implies \chi(\sqrt{m}) = \pm i.$$
We claim that χ^s/χ is quadratic. Indeed, it follows from Lemma 2.3.6 and the fact that τ is irreducible that χ^s/χ is quadratic on inertia. We now compute $(\chi^s/\chi)(\pi)$, where $\pi = 1 - \sqrt{m}$ is a uniformizer:
$$\begin{aligned}(\chi^s/\chi)(\pi) &= \chi(s(\pi)/\pi) = \chi((1+\sqrt{m})/(1-\sqrt{m})) = \chi(u(1-\sqrt{m})/(1-\sqrt{m})) \\ &= \chi(u),\end{aligned}$$
where $u = -(2-\sqrt{m})/3 \in \mathcal{O}_K^\times$. We have $u^{-1} = \sqrt{m}^3 \cdot (2\sqrt{m}-1)$, as elements of $(\mathcal{O}_K/\mathfrak{p}^6)^\times/U_\mathfrak{f}$, hence
$$\chi(u^{-1}) = \chi(\sqrt{m})^3 \cdot \chi(2\sqrt{m}-1) = (\pm i)^3(\pm i) = \pm 1,$$
showing that χ^s/χ is quadratic, as claimed.

 Since χ^s/χ is quadratic, it factors via the norm map by Corollary 2.3.4. So ρ_E is triply imprimitive by Proposition 2.3.12. This leads to a contradiction as in the proof of Proposition 6.3.1. We conclude there are no types arising from an elliptic curve for $k = 6$ and $d = -20$.

This completes the proof by cases. □

τ	e	condexp(τ)	$\mathrm{Nm}_{L\|K}(\mathcal{O}_L^\times)/U_\mathfrak{f}$	L'	$\mathrm{Gal}(L\,\|\,\mathbb{Q}_2)$	E
trivial	1	0	–	2.1.0.1	1T1 $\simeq C_1$	11a1
ε_{-1}	2	4	–	2.2.2.1	2T1 $\simeq C_2$	176b2
ε_2	2	6	–	2.2.3.1	2T1 $\simeq C_2$	704a2
ε_{-2}	2	6	–	2.2.3.3	2T1 $\simeq C_2$	704k2
$\tau_{\mathrm{sc},2}(5,1,3)$	3	2	trivial	2.3.2.1	3T2 $\simeq S_3$	20a1
$\tau_{\mathrm{sc},2}(5,4,4)$	4	8	$\langle(1,4)\rangle \simeq \mathbb{Z}/6$	2.4.11.18	4T3 $\simeq D_4$	256a1
$\tau_{\mathrm{sc},2}(5,4,4)\otimes\varepsilon_{-4}$	4	8	$\langle(1,10)\rangle \simeq \mathbb{Z}/6$	2.4.11.17	4T3 $\simeq D_4$	256d1
$\tau_{\mathrm{ps},2}(1,4,4)$	4	8	$\langle(1,0)\rangle \simeq \mathbb{Z}/2$	2.4.11.1	4T1 $\simeq C_4$	768b1
$\tau_{\mathrm{ps},2}(1,4,4)\otimes\varepsilon_{-4}$	4	8	$\langle(1,2)\rangle \simeq \mathbb{Z}/2$	2.4.11.2	4T1 $\simeq C_4$	768h1
$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-4}$	6	4	trivial	2.6.8.1	6T3 $\simeq D_6$	80b1
$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{8}$	6	6	$\langle(0,3)\rangle \simeq \mathbb{Z}/2$	2.6.11.1	6T3 $\simeq D_6$	320c2
$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-8}$	6	6	$\langle(1,0)\rangle \simeq \mathbb{Z}/2$	2.6.11.9	6T3 $\simeq D_6$	320f2
$\tau_{\mathrm{sc},2}(-20,3,4)$	8	5	trivial	2.8.16.65	8T8 $\simeq 2\cdot D_4$	96a1
$\tau_{\mathrm{sc},2}(-4,3,4)$	8	5	trivial	2.8.16.66	8T8 $\simeq 2\cdot D_4$	288a1
$\tau_{\mathrm{sc},2}(-20,3,4)\otimes\varepsilon_8$	8	6	$\langle(2,1)\rangle \simeq \mathbb{Z}/2$	2.8.18.74	8T8 $\simeq 2\cdot D_4$	192a2
$\tau_{\mathrm{sc},2}(-4,3,4)\otimes\varepsilon_8$	8	6	$\langle(2,1)\rangle \simeq \mathbb{Z}/2$	2.8.18.73	8T8 $\simeq 2\cdot D_4$	576f2
$\tau_{\mathrm{sc},2}(-4,6,4)\otimes\varepsilon_8$	8	8	$\langle(3,3)\rangle \simeq \mathbb{Z}/4$	2.8.24.66	8T8 $\simeq 2\cdot D_4$	256b2
$\tau_{\mathrm{sc},2}(-4,6,4)$	8	8	$\langle(3,1)\rangle \simeq \mathbb{Z}/4$	2.8.24.68	8T8 $\simeq 2\cdot D_4$	256c1

Table 10. Types, defining fields, and elliptic curves realizing each nonexceptional inertial type over $\mathbb{Q}_2$ in the case of potentially good reduction, where $2\cdot D_4 \simeq QD_{16}$ is the quasi-dihedral group of order 16

Remark 6.5.2. In Proposition 6.5.1, we always have $e = 8$. To ensure that we do not get the same kind of contradiction that arises in the last case of the proof of this proposition, to show that this type occurs we must check the order of χ^s/χ in all cases. Instead, we will show in Section 6.7 that there is an elliptic curve with that type.

6.6. **Proof of theorem.** We are now ready to prove the main result of Section 6.

Proof of Theorem 6.1.4. Since $e \neq 24$, by Lemma 3.2.5 and Proposition 2.4.1, τ_E is either principal series or nonexceptional supercuspidal.

The case when τ_E is principal series is treated by Proposition 6.2.1.

If τ_E is nonexceptional supercuspidal induced from a quadratic extension K, then by Proposition 6.3.1, K has conductor 1 or 4. The case of conductor 1 (unramified) is covered in Proposition 6.4.1 and Proposition 6.5.1 covers the case of conductor 4. □

6.7. **Explicit realization.** In Table 6.7 we match each nonexceptional inertial type with a representative elliptic curve over $\mathbb{Q}_2$, with code online [18].

Proposition 6.7.1. *For $p = 2$ and each nonexceptional inertial type τ arising from an elliptic curve with additive, potentially good reduction, there is a unique descent L' of the* *inertial field. Moreover, either $L' = L$ is Galois or the compositum $L = \mathbb{Q}_4 L'$ is Galois over $\mathbb{Q}_2$, where $\mathbb{Q}_4$ is the quadratic unramified extension of $\mathbb{Q}_2$.*

Proof. The proof is similar to Proposition 5.3.3. □

With the list of fields in hand, the proof of Table 6.7 is similar to Section 5.3, and we now summarize it. We note however that, in contrast with Section 5.3, in the argument below we do not need the description of the norm groups of all the rows in the table, as many rows are deduced by taking quadratic twists of other rows (for which we do use that information); we decided to include the norm groups in all cases for the sake of completeness.

The curve `11a1` has good reduction at 2 proving the first row and twisting by -1, 2, and -2 yields the next three rows. To complete the correspondence, using norm group computations we will find which type corresponds to each field.

We start with the principal series case. The curve `768b1` over $\mathbb{Q}_2$ has conductor 2^8 and obtain good reduction over a cyclic extensions $L_\tau \supseteq \mathbb{Q}_2$ of degree 4 hence its inertial type τ is a principal series of conductor 2^8 determined by a character $\chi|_{I_2}$ of order 4. Moreover, setting $K = \mathbb{Q}_2$ and $\mathfrak{f} = 2^4$, by local class field theory, we have

$$\ker(\chi^A|_{(\mathcal{O}_K/\mathfrak{f})^\times}) = \mathrm{Nm}_{L_\tau|K}(\mathcal{O}_{L_\tau}^\times) \hookrightarrow (\mathcal{O}_K/\mathfrak{f})^\times,$$

where we used that $U_\mathfrak{f} = 1$ in this case. Comparing the norm group obtained from L_τ to the character defined in the first row of Table 8 we conclude that $\tau = \tau_{\mathrm{ps},2}(1,4,4)$. Twisting `768b1` by -1 shoes that the inertial type of `768h1` is $\tau_{\mathrm{ps},2}(1,4,4) \otimes \varepsilon_{-4}$.

We are now left with the supercuspidal types. Following an argument and calculations analogous to Section 5.3 we compute norm groups of the form $\mathrm{Nm}_{L|K}(\mathcal{O}_L^\times)/U_\mathfrak{f}$ inside $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$ where the latter is given by the group structure and generators in Table 6; these norm groups uniquely identify the field L as an extension of K. Comparing the output of this calculation to the definition of the characters in Table 8 plus taking adequate quadratic twists establishes all rows except those corresponding to `96a1`, `288a1`, `192a2`, and `576f2`; since the latter two curves are quadratic twits by ε_8 of the first two, it suffices to determine the types for `96a1` and `288a1`. There is ambiguity here because the fields of good reduction of these curves both contain $\mathbb{Q}_2(\sqrt{d})$ for $d = -4, -20$ and the two resulting norm groups are $\{0\}$. Finally, an argument analogous to that at the end of Section 5.3 for the types $\tau_{\mathrm{sc},3}(\pm 3, 2, 6)$ completes the proof—here we use that $\mathrm{Gal}(L \,|\, \mathbb{Q}_2) \simeq 2 \cdot D_4$ modulo its center has order 8.

The followiing corollary is analogous to Corollary 5.3.4 and follows from the previous discussion.

Corollary 6.7.2. *Let E be an elliptic curve over $\mathbb{Q}_2$ with potentially good reduction. Assume that E semistability defect $e_E \neq 2$. Then there is a unique field L' in Table 6.7 of minimal degree such that E obtains good reduction over L', and τ_E is given by the type that corresponds to this L'.*

7. Exceptional inertial types for E over $\mathbb{Q}_2$

Finally, we consider exceptional inertial types which arise only for $p = 2$.

7.1. **Setup and result.** Let $r = \pm 1, \pm 2$ and define the following elliptic curves over $\mathbb{Q}_2$

$$E_{1,r}\colon ry^2 = x^3 - 3x - 1 \qquad \text{and} \qquad E_{2,r}\colon ry^2 = x^3 + 3x + 2. \tag{7.1.1}$$

These curves have potentially good reduction with semistability defect $e = 24$. We denote by $\tau_{i,r}$ the inertial type of $E_{i,r}$. We have $N = 0$ for all i, r as above. For $i = 1, 2$ and $r = \pm 1, \pm 2$, let $K_{i,r} := \mathbb{Q}_2(E_{i,r}[3])$, where $E_{i,r}$ is given by (7.1.1).

For reasons that will shortly be clear, we also abbreviate $\tau_{\mathrm{ex},2,1} := \tau_{1,1}$ and $\tau_{\mathrm{ex},2,2} := \tau_{2,1}$. Our final result is as follows.

Theorem 7.1.2. *Let E be an elliptic curve over $\mathbb{Q}_2$ with potentially good reduction, semistability defect $e_E = 24$, conductor N_E and inertial type τ_E. Then one of the following holds.*

(a) *If $N_E = 2^3$, then $\tau_E \simeq \tau_{\mathrm{ex},2,1}$.*
(b) *If $N_E = 2^4$, then $\tau_E \simeq \tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-4}$.*
(c) *If $N_E = 2^6$, then $\tau_E \simeq \tau_{\mathrm{ex},2,1} \otimes \varepsilon_8$ or $\tau_E = \tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-8}$.*
(d) *If $N_E = 2^7$, then τ_E is isomorphic to one of $\tau_{\mathrm{ex},2,2}$, $\tau_{\mathrm{ex},2,2} \otimes \varepsilon_{-4}$, $\tau_{\mathrm{ex},2,2} \otimes \varepsilon_8$ or $\tau_{\mathrm{ex},2,2} \otimes \varepsilon_{-8}$.*

Proof. The fields $K_{i,r}$ give the set of all $\widetilde{S}_4$-extensions of $\mathbb{Q}_2$, where $\widetilde{S}_4 \simeq \mathrm{GL}_2(\mathbb{F}_3)$ is the double cover of S_4: see Bayer–Rio [2, Table 10]). Table 17 lists a polynomial whose splitting field is $K_{i,r}$ for each i and r.

Let $K = \mathbb{Q}_2(E[3])$, $G = \mathrm{Gal}(K \,|\, \mathbb{Q}_2)$ and $L = \mathbb{Q}_2^{\mathrm{un}} K$ be the inertial field of E. From the proof of Lemma 3.2.5, we know that $G \simeq \widetilde{S}_4$ a double cover of $\mathrm{P}(\overline{\rho}_{E,3}) \simeq S_4$. Therefore, there is a choice of i, r such that both τ and $\tau_{i,r}$ fix the extension $L \supset \mathbb{Q}_2^{\mathrm{un}}$. We have $\mathrm{Gal}(L \,|\, \mathbb{Q}_2^{\mathrm{un}}) \simeq \Phi \simeq \mathrm{SL}_2(\mathbb{F}_3)$ by Lemma 3.2.1, and since there is only one irreducible $\mathrm{GL}_2(\mathbb{C})$-representation of $\mathrm{SL}_2(\mathbb{F}_3)$ whose image is contained in $\mathrm{SL}_2(\mathbb{C})$, we conclude that $\tau \simeq \tau_{i,r}$.

Note that, for $r = -1, 2, -2$, the curve $E_{1,r}$ is the quadratic twist of $E_{1,1}$ by $-4, 8, -8$, respectively, therefore $\tau_{1,-1} \simeq \tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-4}$, $\tau_{1,2} \simeq \tau_{\mathrm{ex},2,1} \otimes \varepsilon_8$ and $\tau_{1,-2} \simeq \tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-8}$. Similarly, we obtain $\tau_{2,-1} \simeq \tau_{\mathrm{ex},2,2} \otimes \varepsilon_{-4}$, $\tau_{2,2} \simeq \tau_{\mathrm{ex},2,2} \otimes \varepsilon_8$ and $\tau_{2,-2} \simeq \tau_{\mathrm{ex},2,2} \otimes \varepsilon_{-8}$.

Finally, observe that the conductor of $E_{1,1}$ is 2^3 and that of $E_{2,1}$ is 2^7, thus the eight types split in the 4 cases of the theorem according to their conductors. $\square$

7.2. **Explicit characters.** Recall that, as in previous sections, we aim for an explicit description of the types in Theorem 7.1.2 in terms of characters. As explained in Section 2.3, an exceptional type is determined by a triple (L, M, χ), where $L \,|\, \mathbb{Q}_2$ is a cubic extension, $M \,|\, L$ a quadratic extension and $\chi\colon W_M \to \mathbb{C}^\times$ a character such that $\chi \neq \chi^s$ where s is conjugation on $M \,|\, L$. Furthermore, we only need to specify χ on I_M.

For the rest of this section, we write $F_2 := \mathbb{Q}_2(\sqrt[3]{2})$.

Lemma 7.2.1. *We keep the above notations. All fields $K_{i,r}$ contain the cubic field F_2, and the unique unramified quadratic extension $\mathbb{Q}_4(\sqrt[3]{2})$ of F_2. Moreover:*

(a) *The fields $K_{1,r}$, $r = \pm 1, \pm 2$, contain the two ramified quadratic extensions $\mathbb{Q}_2(\sqrt[3]{2}, \alpha_\pm)$ of F_2, where α_- is a root of $x^2 + \sqrt[3]{2}x + \sqrt[3]{2}$ and α_+ a root of $x^2 + \sqrt[3]{2}x + \sqrt[3]{2}^2 + \sqrt[3]{2}$.*
(b) *The fields $K_{2,r}$, $r = \pm 1, \pm 2$, contain the two ramified quadratic extensions $\mathbb{Q}_2(\sqrt[3]{2}, \beta_\pm)$ of F_2, where β_- is a root of $x^2 + 2x + \sqrt[3]{2} + 2$ and β_+ a root of $x^2 + 2x + \sqrt[3]{2} + 6$.*

Proof. Direct calculation with subfields, see code [18]. $\square$

Lemma 7.2.2. *Let $K = \mathbb{Q}_4(\sqrt[3]{2})$ or $K = K_{i,r}$ be one of the extensions given in Lemma 7.2.1. Then, Tables 11, 12, 13 and 14 give the structure and generators for the three groups $U_{\mathfrak{f}}$, $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$, and $(\mathbb{Z}_2[\sqrt[3]{2}]/\mathfrak{q})^\times/U_{\mathfrak{f}}$.*

Proof. The induction starts to get quite tricky, so we now only claim the content of the tables up to a finite conductor exponent bound for which an explicit Magma calculation gives the result [18]. (In principal, the general statement follows by the same kind of checks as in Lemma 5.1.2.) □

Proposition 7.2.3. *Let τ be an exceptional type arising from E over $\mathbb{Q}_2$. Then $\tau|_{I_{F_2}}$ is one of the (non-exceptional) supercuspidal types listed in Table 15.*

Proof. Let $\tau := \rho_E|_{I_2}$ be an exceptional type arising from E over $\mathbb{Q}_2$. From Theorem 7.1.2 and its proof we know τ corresponds to a field $L := K_{i,r}$ for some i, r. Since F_2 and its two conjugated extensions are the unique cubic extension of $\mathbb{Q}_2$ inside all the $K_{i,r}$, we conclude that $\tau|_{I_{F_2}}$ is imprimitive by Proposition 2.3.13. Furthermore, $\tau|_{I_{F_2}}$ has conductor exponent 5, 8, 14, or 17. Let $K \mid F_2$ be the quadratic extension from which $\tau|_{I_{F_2}}$ is induced. Let $\chi\colon W_K \to \mathbb{C}^\times$ be the character such that

$$\tau|_{I_{F_2}} = \rho_E|_{I_{F_2}}$$

where

$$\rho_E|_{G_{F_2}} := \operatorname{Ind}_K^{F_2} \chi.$$

By the conductor exponent formula (2.3.5), χ has conductor exponent 3, 6, 11, or 12. We also know that

$$\ker \chi^{\mathrm{A}} = \mathrm{Nm}_{L|K}(\mathcal{O}_L^\times)/U_{\mathfrak{f}} \hookrightarrow (\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}.$$

In Table 16, we compute the norm group $\mathrm{Nm}_{L|K}(\mathcal{O}_L^\times)/U_{\mathfrak{f}}$ for each possible L and each of the ramified quadratic extensions $K \mid F_2$ listed in Lemma 7.2.1 (code at [18]). Since the projective image of $\mathrm{P}(\rho_E|_{G_{F_2}})$ is isomorphic to D_4, we conclude that χ is of order 4 on I_K. Now, using all the previous constraints we determine the *unique* quadratic extension $K \mid F_2$ from which $\tau|_{I_{F_2}}$ is induced, together with the corresponding character χ. Indeed, the order of the norm groups suffices to decide what is the correct quadratic extension in all cases. For the correct quadratic extension $K \mid F_2$, Table 16 also gives the generators of $\ker \chi^{\mathrm{A}}$ in terms of those of the group $(\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$ listed in Table 13. Table 15 contains, up to complex conjugation, all the inertial types satisfying those conditions. Therefore, $\tau|_{I_{F_2}}$ must be one the types listed in it. □

7.3. **Explicit realization.** In Table 17 we give all exceptional inertial types for elliptic curves over $\mathbb{Q}_2$ with potentially good reduction together with a curve realizing each type. In particular, this shows that all the possible types over $\mathbb{Q}_2$ we compute indeed arise from elliptic curves. A similar corollary holds as Corollary 5.3.4.

References

[1] Alexander J. Barrios and Manami Roy, *Representations attached to elliptic curves with a non-trivial odd torsion point*, Bull. Lond. Math. Soc. **54** (2022), no. 5, 1846–1861. 1.2

[2] Pilar Bayer and Anna Rio, *Dyadic exercises for octahedral extensions*, J. Reine Angew. Math. **517** (1999), 1–17. 7.1

$K \mid F_2$	$\mathfrak{f}$	f	$U_{\mathfrak{f}}$
$\mathbb{Q}_4(\sqrt[3]{2})$	$\mathfrak{p}^f$	1	trivial
		2	$\langle u_4\rangle \simeq \mathbb{Z}/2$
		3	$\langle u_4\rangle \simeq \mathbb{Z}/4$
		4	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$
		5	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/8$
		6	$\langle u_2\rangle \times \langle _3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		7	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		8	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/16$
		9	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/16$
$\mathbb{Q}_2(\sqrt[3]{2}, \alpha_\pm)$	$\mathfrak{p}^f$	$1, 2, 3, 4$	trivial
		$5, 6$	$\langle u_4\rangle \simeq \mathbb{Z}/2$
		$7, 8$	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$
		$9, 10$	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$
		$11, 12$	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$
		$13, 14$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$
		$15, 16$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		$17, 18$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/8$
$\mathbb{Q}_2(\sqrt[3]{2}, \beta_\pm)$	$\mathfrak{p}^f$	$1, 2$	trivial
		$3, 4$	$\langle u_4\rangle \simeq \mathbb{Z}/2$
		$5, 6$	$\langle u_4\rangle \simeq \mathbb{Z}/4$
		$7, 8$	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$
		$9, 10, 11, 12$	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/8$
		$13, 14$	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		$15, 16$	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/16$
		$17, 18$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/16$
		$19, 20$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/16$
		$21, 22$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/32$
		$23, 24$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/32$
		$25, 26$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/8 \times \mathbb{Z}/32$
		$27, 28$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/8 \times \mathbb{Z}/64$

TABLE 11. Group structure of $U_{\mathfrak{f}}$ for the quadratic extensions $K \mid F_2$ contained in $K_{1,r}$ and $K_{2,r}$, with $r = \pm 1, \pm 2$

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K	$\mathfrak{f}$	f	$(\mathbb{Z}_2[\sqrt[3]{2}]/\mathfrak{q})^\times/U_\mathfrak{f}$
$\mathbb{Q}_4(\sqrt[3]{2})$	$\mathfrak{p}^f$	≥ 1	trivial
$\mathbb{Q}_2(\sqrt[3]{2},\alpha)$	$\mathfrak{p}^f$	$1,2$	trivial
$\mathbb{Q}_2(\sqrt[3]{2},\beta)$	$\mathfrak{p}^f$	$1,2,3,4,5,6,7,8,9,10$	trivial

TABLE 12. Group structure of $(\mathbb{Z}_2[\sqrt[3]{2}]/\mathfrak{q})^\times/U_\mathfrak{f}$ for the quadratic extensions $K\,|\,F_2$ contained in $K_{1,r}$ and $K_{2,r}$, with $r=\pm1,\pm2$

Generators	$\mathbb{Q}_4(\sqrt[3]{2})=\mathbb{Q}_2(\sqrt[3]{2},\nu)$
u_1	$-2\nu-1$
u_2	$(-2\nu-1)\sqrt[3]{2}^2+(2\nu+1)\sqrt[3]{2}-2\nu-7$
u_3	$\sqrt[3]{2}^2-2\nu-1$
u_4	$(-2\nu-1)\sqrt[3]{2}^2+(2\nu+1)\sqrt[3]{2}+\nu-1$
	$\mathbb{Q}_2(\sqrt[3]{2},\alpha_\pm)$
u_1	$\sqrt[3]{2}^2+\sqrt[3]{2}+1$
u_2	$-\sqrt[3]{2}^2\alpha_\pm-1$
u_3	$(-2\sqrt[3]{2}^2+4)\alpha_\pm-\sqrt[3]{2}^2-\sqrt[3]{2}+3$
u_4	$(-\sqrt[3]{2}^2+2)\alpha_\pm-\sqrt[3]{2}^2-\sqrt[3]{2}+1$
u_5	$-\alpha_\pm+1$
	$\mathbb{Q}_2(\sqrt[3]{2},\beta_\pm)$
u_1	$\sqrt[3]{2}^2+1$
u_2	$-\beta_\pm-1$
u_3	$(-2\sqrt[3]{2}+2)\beta_\pm-\sqrt[3]{2}^2+1$
u_4	$(-\sqrt[3]{2}+1)\beta_\pm+\sqrt[3]{2}^2-\sqrt[3]{2}+1$
u_5	$(-\sqrt[3]{2}+1)\beta_\pm-2\sqrt[3]{2}+1$

TABLE 13. Generators for the group $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$ for the quadratic extensions $K\,|\,F_2$ contained in $K_{1,r}$ and $K_{2,r}$, with $r=\pm1,\pm2$; here, $\nu^2+\nu+1=0$

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$K \mid F_2$	$\mathfrak{f}$	f	$(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$
		1	$\langle u_4\rangle \simeq \mathbb{Z}/3$
		2	$\langle u_4\rangle \simeq \mathbb{Z}/6$
		3	$\langle u_4\rangle \simeq \mathbb{Z}/12$
$\mathbb{Q}_4(\sqrt[3]{2})$	$\mathfrak{p}^f$	4	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/12$
		5	$\langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/24$
		6	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/24$
		≥ 7	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \simeq$ $\mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor \frac{f-7}{3} \rfloor+1} \times \mathbb{Z}/2^{\lfloor \frac{f-6}{3} \rfloor+1} \times \mathbb{Z}/(3 \cdot 2^{\lfloor \frac{f-5}{3} \rfloor+3})$
		1	trivial
		2	$\langle u_5\rangle \simeq \mathbb{Z}/2$
		3	$\langle u_5\rangle \simeq \mathbb{Z}/4$
		$4,5$	$\langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$
		$6,7$	$\langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$
$\mathbb{Q}_2(\sqrt[3]{2}, \alpha_\pm)$	$\mathfrak{p}^f$	$8,9$	$\langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		$10,11$	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		$12,13$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/8$
		$14,15$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/16$
		$16,17$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/16$
		≥ 18	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq$ $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor \frac{f-18}{6} \rfloor+2} \times \mathbb{Z}/2^{\lfloor \frac{f-16}{6} \rfloor+2} \times \mathbb{Z}/2^{\lfloor \frac{f-14}{6} \rfloor+4}$
		1	trivial
		$2,3$	$\langle u_5\rangle \simeq \mathbb{Z}/2$
		$4,5$	$\langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$
		$6,7$	$\langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$
		$8,9$	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$
		10	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$
$\mathbb{Q}_2(\sqrt[3]{2}, \beta_\pm)$	$\mathfrak{p}^f$	11	$\langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4$
		$12,13$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4$
		$14,15$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/4$
		$16,17$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/8$
		$18,19$	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/8 \times \mathbb{Z}/8$
		≥ 20	$\langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \simeq$ $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor \frac{f-20}{6} \rfloor+3} \times \mathbb{Z}/2^{\lfloor \frac{f-18}{6} \rfloor+3} \times \mathbb{Z}/2^{\lfloor \frac{f-16}{6} \rfloor+3}$

TABLE 14. Group structure of $(\mathcal{O}_K/\mathfrak{f})^\times/U_\mathfrak{f}$ for the quadratic extensions $K \mid F_2$ contained in $K_{1,r}$ and $K_{2,r}$, with $r = \pm 1, \pm 2$

K	$\mathfrak{f}$	f	r	values of χ on generators	$\tau\|_{F_2}$	$\operatorname{condexp}(\tau\|_{F_2})$
$\mathbb{Q}_2(\sqrt[3]{2},\alpha_+)$	$\mathfrak{p}^f$	3	4	i	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\alpha_+),3,4)$	5
		6	4	$-1,\ 1,\ i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\alpha_+),6,4)$	8
		12	4	$-1,-1,\ 1,\ 1,\ i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\alpha_+),12,4)$	14
		12	4	$-1,\ 1,\ 1,\ -1,\ i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\alpha_+),12,4)$	14
$\mathbb{Q}_2(\sqrt[3]{2},\beta_+)$	$\mathfrak{p}^f$	11	4	$-1,\ 1,\ -i,\ i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\beta_+),11,4)$	17
		11	4	$1,\ 1,\ i,\ -i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\beta_+),11,4)$	17
		11	4	$1,\ -1,\ -i,\ i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\beta_+),11,4)$	17
		11	4	$-1,\ -1,\ -i,\ i$	$\tau_{\mathrm{sc},\mathfrak{p}}(\mathbb{Q}_2(\sqrt[3]{2},\beta_+),11,4)$	17

TABLE 15. Restrictions of primitive inertial types from $\mathbb{Q}_2$ to F_2

E	f	$K \mid F_2$	$\mathrm{Nm}_{L\|K}(\mathcal{O}_L^\times)/U_{\mathfrak{f}} \le (\mathcal{O}_K/\mathfrak{f})^\times/U_{\mathfrak{f}}$	$\operatorname{condexp}(\tau_E\|_{F_2})$
$E_{1,1}$	3	$\mathbb{Q}_2(\sqrt[3]{2},\alpha_-)$	$\mathbb{Z}/2$	5
		$\mathbb{Q}_2(\sqrt[3]{2},\alpha_+)$	trivial	
$E_{1,-1}$	6	$\mathbb{Q}_2(\sqrt[3]{2},\alpha_-)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$	8
		$\mathbb{Q}_2(\sqrt[3]{2},\alpha_+)$	$\langle u_3u_4u_5^2, u_4\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2$	
$E_{1,2}$	12	$\mathbb{Q}_2(\sqrt[3]{2},\alpha_-)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	14
		$\mathbb{Q}_2(\sqrt[3]{2},\alpha_+)$	$\langle u_1u_2u_4, u_1u_2u_5^4, u_1u_2u_3u_5^4, u_2u_5^6\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	
$E_{1,-2}$	12	$\mathbb{Q}_2(\sqrt[3]{2},\alpha_-)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	14
		$\mathbb{Q}_2(\sqrt[3]{2},\alpha_+)$	$\langle u_1u_2u_4, u_2u_5^4, u_2u_3u_5^4, u_1u_3u_5^2\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	
$E_{2,1}$	11	$\mathbb{Q}_2(\sqrt[3]{2},\beta_-)$	$\langle u_3u_4^2u_5^2, u_2u_4^2, u_3u_4u_5^3\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	17
		$\mathbb{Q}_2(\sqrt[3]{2},\beta_+)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	
$E_{2,-1}$	11	$\mathbb{Q}_2(\sqrt[3]{2},\beta_-)$	$\langle u_2, u_3u_4^2u_5^2, u_4u_5\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	17
		$\mathbb{Q}_2(\sqrt[3]{2},\beta_+)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	
$E_{2,2}$	11	$\mathbb{Q}_2(\sqrt[3]{2},\beta_-)$	$\langle u_3u_5^2, u_2, u_3u_4u_5^3\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	17
		$\mathbb{Q}_2(\sqrt[3]{2},\beta_+)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	
$E_{2,-2}$	11	$\mathbb{Q}_2(\sqrt[3]{2},\beta_-)$	$\langle u_3u_5^2, u_2u_4^2, u_2u_3u_4u_5^3\rangle \simeq \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	17
		$\mathbb{Q}_2(\sqrt[3]{2},\beta_+)$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$	

TABLE 16. Norm groups $\mathrm{Nm}_{L|K}(\mathcal{O}_L^\times)/U_{\mathfrak{f}}$ and kernels $\ker \chi^{\mathrm{A}}$ of the inducing characters for the restrictions of exceptional types to F_2.

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τ	e	condexp(τ)	L'	$\mathrm{Gal}(L \mid \mathbb{Q}_2)$	E
$\tau_{\mathrm{ex},2,1}$	24	3	2.8.10.2	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	648b1
$\tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-4}$	24	4	2.8.12.29	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	1296c1
$\tau_{\mathrm{ex},2,1} \otimes \varepsilon_{8}$	24	6	2.8.16.73	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	5184e1
$\tau_{\mathrm{ex},2,1} \otimes \varepsilon_{-8}$	24	6	2.8.16.71	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	5184w1
$\tau_{\mathrm{ex},2,2}$	24	7	2.8.22.132	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	3456a1
$\tau_{\mathrm{ex},2,2} \otimes \varepsilon_{-4}$	24	7	2.8.22.136	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	3456e1
$\tau_{\mathrm{ex},2,2} \otimes \varepsilon_{8}$	24	7	2.8.22.138	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	3456o1
$\tau_{\mathrm{ex},2,2} \otimes \varepsilon_{-8}$	24	7	2.8.22.137	8T23 $\simeq \mathrm{GL}_2(\mathbb{F}_3)$	3456c1

TABLE 17. Elliptic curves for each exceptional inertial type over $\mathbb{Q}_2$.

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Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK
Email address: lassina.dembele@kcl.ac.uk

Instituto de Ciencias Matemáticas, CSIC, Calle Nicolás Cabrera 13–15, 28049 Madrid, Spain
Email address: nuno.freitas@icmat.es

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA; School of Mathematics and Statistics, Carslaw Building (F07), University of Sydney, NSW 2006, Australia
Email address: jvoight@gmail.com
URL: http://jvoight.github.io/