AN ORTHOGONAL PERSPECTIVE ON GAUSS COMPOSITION

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Abstract. We revisit Gauss composition over a general base scheme, with a focus on orthogonal groups. We show that the Clifford and norm functors provide a discriminant-preserving equivalence of categories between binary quadratic modules and pseudoregular modules over quadratic algebras. This perspective synthesizes the constructions of Kneser and Wood, reconciling algebraic and geometric approaches and clarifying the role of orientations and the natural emergence of narrow class groups. As an application, we restrict to lattices and show that binary orthogonal eigenforms correspond to Hecke characters.

Contents

1. Introduction

1.1. **Motivation.** *Gauss composition* is the term commonly employed for the bijection between equivalence classes of binary quadratic forms over $\mathbb{Z}$ and class groups of quadratic orders. Many papers have been written providing generalizations of this all important result: we provide in section 7.3 an overview of recent work.

In this paper, we view Gauss composition within the framework of orthogonal groups, synthesizing and extending previous work. We show that Gauss composition over an arbitrary base can be understood through Clifford algebras with inverse given by the norm, recovering both the classical approach of Kneser [Kne82] and the more geometric approach of Wood [Woo11]. Organized by orthogonal groups, this perspective also naturally reconciles notions of orientations and morphisms, thereby providing a general composition law (including for narrow class groups) as well as applications to spaces of binary orthogonal modular forms.

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1.2. **Setup.** To state our results, we will first need some definitions and notation; for further details in the setup, see section 2.

Let X be a scheme. A *quadratic $\mathscr{O}_X$-module* is a locally free $\mathscr{O}_X$-module $\mathscr{M}$ of finite rank equipped with a quadratic map $Q\colon \mathscr{M} \to \mathscr{L}$ taking values in an invertible $\mathscr{O}_X$-module, which we call the *value line bundle*. The quadratic map is equivalently specified by a global section of $(\operatorname{Sym}_2 \mathscr{M})^\vee \otimes \mathscr{L}$. A *similarity* between two quadratic $\mathscr{O}_X$-modules is the natural commutative square (2.1.2).

The Clifford functor associates to a quadratic $\mathscr{O}_X$-module $Q\colon \mathscr{M} \to \mathscr{L}$ the odd Clifford bimodule $\operatorname{Clf}^1(Q)$ over the even Clifford $\mathscr{O}_X$-algebra $\operatorname{Clf}^0(Q)$. When $\mathscr{M}$ is binary, $\operatorname{Clf}^0(Q)$ is a locally free quadratic $\mathscr{O}_X$-algebra and the Clifford functor fibers over the category **Quad** of locally free quadratic algebras (itself fibered over **Sch**, the category of schemes).

The Clifford bimodule inherits the following key structural property. Let $Y \to X$ be a locally free morphism of degree $n \geq 1$ and let $\mathscr{I}$ be an $\mathscr{O}_Y$-module which is locally free of rank n as an $\mathscr{O}_X$-module. Recall that the *regular* $\mathscr{O}_Y$-module is $\mathscr{O}_Y$ itself (considered as an $\mathscr{O}_Y$-module). Accordingly, we say that $\mathscr{I}$ is *pseudoregular* if for all open $U \subseteq X$ and all $\alpha \in \mathscr{O}_Y(U)$, the characteristic polynomial of α acting by multiplication on $\mathscr{O}_Y(U)$ and $\mathscr{I}(U)$ are equal. When $Y \to X$ has degree $n = 2$, it is enough to check traces (Equation (3.2.3)), and so $\mathscr{I}$ is also called *traceable* by Wood [Woo11, p. 1758].

Let $Q\colon \mathscr{M} \to \mathscr{L}$ be a binary quadratic $\mathscr{O}_X$-module and let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra. An *$\mathscr{O}_Y$-orientation* of Q is an $\mathscr{O}_X$-algebra isomorphism $\operatorname{Clf}^0(Q) \xrightarrow{\sim} \mathscr{O}_Y$. A similarity preserving orientations is called an *oriented similarity*. For further discussion on orientations, see below and section 1.7.

Norms furnish an inverse to the Clifford functor, as follows. Let $\mathscr{I}$ be a $\mathscr{O}_Y$-module which is locally free of rank 2 as an $\mathscr{O}_X$-module. We define a canonical *norm map*

$$N_{\mathscr{I}}\colon \mathscr{I} \to \mathscr{N}(\mathscr{I}) \tag{1.2.1}$$

where

$$\mathscr{N}(\mathscr{I}) := \bigwedge^2 \mathscr{I} \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee; \tag{1.2.2}$$

it earns its name from the property that

$$N_{\mathscr{I}}(\gamma x) = \operatorname{Nm}(\gamma) N_{\mathscr{I}}(x) \tag{1.2.3}$$

for all $x \in \mathscr{I}(U)$ and $\gamma \in \mathscr{O}_Y(U)$ on open $U \subseteq X$, matching Kneser [Kne82, section 4]. The norm map arises from a twist of the *canonical exterior form*, the unique quadratic map

$$E_{\mathscr{I}}\colon \mathscr{I} \otimes \mathscr{O}_Y/\mathscr{O}_X \to \bigwedge^2 \mathscr{I} \otimes \mathscr{O}_Y/\mathscr{O}_X \tag{1.2.4}$$

such that $E_{\mathscr{I}}(x \otimes \gamma) = (\gamma x \wedge x) \otimes \gamma$ for all $x \in \mathscr{I}(U)$ and $\gamma \in (\mathscr{O}_Y/\mathscr{O}_X)(U)$. After appropriate identifications and (perhaps a bit surprisingly) a duality, we show this agrees with the construction of Wood [Woo11, Theorem 2.1] in section 5.3.

1.3. **Similarity classes.** Our first theorem corresponds to the reductive group GSO.

Theorem 1.3.1. *The Clifford and norm functors define a discriminant-preserving equivalence of categories fibered over* **Quad** *between*

$\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules under oriented similarities

and

pseudoregular $\mathscr{O}_Y$-modules under $\mathscr{O}_Y$-module isomorphisms.

In particular, the Clifford and norm are quasi-inverse functors; and if $\mathscr{M} \leftrightarrow \mathscr{I}$ in the equivalence then we have an isomorphism of group schemes $\mathrm{GSO}(\mathscr{M}) \simeq \mathrm{Aut}_{\mathscr{O}_Y}(\mathscr{I})$ over X. Note that free objects correspond to each other in the two categories above, indeed there are universal framed objects.

We say a quadratic $\mathscr{O}_X$-module is *primitive* if the values of Q generate $\mathscr{L}$ on every open. The restriction to primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules under oriented similarities gives an equivalence to invertible (i.e., locally principal) $\mathscr{O}_Y$-modules under $\mathscr{O}_Y$-module isomorphisms (Equation (6.2.2)): in particular, the isomorphism class of $\mathrm{Clf}^1(Q)$ as a $\mathscr{O}_Y$-module gives a well-defined element of $\mathrm{Pic}(Y)$. We obtain following corollary.

Corollary 1.3.2. *Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra. Then the Clifford and norm functors define mutually inverse bijections between the set of oriented similarity classes of primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules and the group* $\operatorname{Pic} Y$.

Equation (1.3.2) provides, for each $\mathscr{O}_Y$, a composition law on the set of oriented similarity classes of primitive binary quadratic modules. Gauss composition becomes the special case $X = \operatorname{Spec} \mathbb{Z}$: the choice of orientation is implicit in the ordered choice of basis in which the binary quadratic form is expressed, and treats the real and imaginary cases uniformly (see Equation (6.2.3)).

Although Wood [Woo11] did not make it explicit, other authors [O'Do16, Dal24, MB2025] have adapted her equivalence of categories to provide a composition law as in Equation (1.3.2) using a notion of orientation that differs from ours: see the next section and section 1.7 for a comparison.

Forgetting orientations, we have the following theorem, corresponding to the reductive group GO.

Corollary 1.3.3. *The Clifford and norm functors define a discriminant-preserving equivalence of categories between*

binary quadratic $\mathscr{O}_X$-modules under similarities

and

pseudoregular modules over quadratic $\mathscr{O}_X$-algebras under isomorphisms.

In particular, they furnish mutually inverse bijections between the set of similarity classes of primitive binary quadratic $\mathscr{O}_X$-modules and $\mathrm{Pic}(Y)/\mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$.

Mondal–Venkata Balaji [MB2025, Theorem 3.1, Theorem 5.11] also prove a version of Equation (1.3.3) (bijections without the equivalence of categories), and instead of constructing an explicit norm functor they apply the theory of *universal norms*. Classically, Equation (1.3.3) describes $\mathrm{GL}_2(\mathbb{Z})$-equivalence, whose classes correspond to the quotient of the class group by inversion. Unfortunately, having lost the orientation, we lose the group structure and therefore the composition law.

1.4. **Similitude classes.** In the study of quadratic modules, often the value line bundle is fixed. For example, the more basic notion of a *quadratic form* corresponds to the case where the value line bundle is trivial, $\mathscr{L} = \mathscr{O}_X$.

Accordingly, we define a *similitude* between $Q\colon \mathscr{M} \to \mathscr{L}$ and $Q'\colon \mathscr{M}' \to \mathscr{L}'$ to be a similarity with $\mathscr{L} = \mathscr{L}'$; its *similitude factor* is $\lambda \in \mathscr{O}_X(X)^\times$. In this way, the similitude class of a binary quadratic module remembers the value line bundle (not just up to isomorphism);

therefore the category of binary quadratic modules under similitudes is fibered over **Pic**, the category of invertible modules over schemes under isomorphisms.

The rigidification provided by this additional data is accounted for in the Clifford functor, as follows: a ($\mathscr{L}$-)*rigidification* of $\mathscr{I}$ is an isomorphism $r\colon \mathscr{N}(\mathscr{I}) \xrightarrow{\sim} \mathscr{L}$. Wood [Woo11, section 5] calls this an *$\mathscr{L}$-type orientation*, but we avoid that name to distinguish it from the orientation defined above. Since $\mathscr{N}(\mathscr{I}) = \bigwedge^2 \mathscr{I} \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee$, given an $\mathscr{O}_Y$-orientation, a $\mathscr{L}$-rigidification of $\mathscr{I}$ canonically determines an isomorphism $\bigwedge^2 \mathscr{I} \xrightarrow{\sim} \mathscr{L} \otimes (\mathscr{O}_Y/\mathscr{O}_X)$ and conversely. From a rigidification r of $\mathscr{I}$, we define the *modified norm map* by $r \circ N_{\mathscr{I}}$.

Theorem 1.4.1. *The Clifford and modified norm functors define a discriminant-preserving equivalence of categories fibered over* $\mathbf{Quad} \times_{\mathbf{Sch}} \mathbf{Pic}$ *between*

$\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules under oriented similitudes

and

rigidified pseudoregular $\mathscr{O}_Y$-modules under $\mathscr{O}_Y$-module isomorphisms

Corollary 1.4.2. *The Clifford and modified norm functors furnish mutually inverse bijections between the set of similitude classes of primitive binary quadratic $\mathscr{O}_X$-modules with value line bundle $\mathscr{L}$ and the fiber of the map*

$$\begin{aligned} \operatorname{Pic} Y &\to \operatorname{Pic} X \\ \mathscr{I} &\mapsto \mathscr{N}(\mathscr{I}) \end{aligned} \tag{1.4.3}$$

over $[\mathscr{L}] \in \operatorname{Pic} X$.

Similarly, as in Equation (1.3.3) we may forget orientations, corresponding to the further quotient by $\operatorname{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$ in Equation (1.4.1).

1.5. **Isometry classes.** Having fixed the line bundle, we may now restrict similarities according to their similitude factor; this opens the door to generalizing a key aspect of Gauss composition, where $\mathrm{SL}_2(\mathbb{Z})$-classes of *indefinite* binary quadratic forms are in bijection with the *narrow* class group of a real quadratic field. This is nicely explained by restricting the group of similitude factors, leading to the orthogonal groups SO and O.

An *isometry* is a similitude with similitude factor $\lambda = 1$. If $\mathscr{I}, \mathscr{I}'$ are rigidified $\mathscr{O}_Y$-modules, we say that an $\mathscr{O}_Y$-module isomorphism $\varphi\colon \mathscr{I} \to \mathscr{I}'$ is *rigidified* if the induced isomorphism $\mathscr{N}(\mathscr{I}) \to \mathscr{N}(\mathscr{I}')$ commutes with the rigidifications.

Theorem 1.5.1. *The Clifford and norm functors define a discriminant-preserving equivalence of categories fibered over* $\mathbf{Quad} \times_{\mathbf{Sch}} \mathbf{Pic}$ *between*

$\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules under oriented isometries

and

rigidified pseudoregular $\mathscr{O}_Y$-modules under rigidified $\mathscr{O}_Y$-module isomorphisms.

These similarly yield corollaries in the vein of Gauss composition, as follows. Over a fixed base, we may work a bit more generally. More generally, let $H \leq \mathscr{O}_X(X)^\times$ be a subgroup; an *H-similitude* is a similitude whose similitude factor lies in H. There is a canonical isomorphism $\operatorname{Pic} Y \cong H^1(Y, \mathscr{O}_Y^\times)$. Let $\mathscr{O}_Y^1 := \ker(\mathrm{Nm}\colon \mathscr{O}_Y^\times \to \mathscr{O}_X^\times)$. Considering $H \leq \mathscr{O}_X(X)^\times$ as a constant subsheaf, let $\mathscr{O}_Y^{(H)} := \mathrm{Nm}^{-1}(H)$ and define $\operatorname{Pic}^{(H)} Y := H^1(Y, \mathscr{O}_Y^{(H)})$. The map $\mathscr{O}_Y^{(H)} \to \mathscr{O}_Y^\times$ induces a natural map $\operatorname{Pic}^{(H)} Y \to \operatorname{Pic} Y$.

Corollary 1.5.2. *Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra. Then for all $H \leq \mathscr{O}_X(X)^\times$, the Clifford and norm functors define bijections between the set of oriented H-similitude classes of primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules and the group* $\mathrm{Pic}^{(H)} Y$.

For example, if $K \supseteq F$ is a quadratic extension of number fields with corresponding maps on integral spectra $Y = \operatorname{Spec} \mathbb{Z}_K \to X = \operatorname{Spec} \mathbb{Z}_F$, then we may take $H = \mathbb{Z}_{F,>0}^\times$ to be the subgroup of units that are positive at all real places of F, and then $\mathrm{Pic}^{(H)}(Y)$ is the narrow class group of $\mathbb{Z}_K$. In particular, if $F = \mathbb{Q}$ and K is real quadratic, then we recover Gauss composition for the narrow class group (see eq. (6.2.4)).

Of course we have similar notions where we forget $\mathscr{O}_Y$-orientations, corresponding to the group O.

1.6. **Lattices.** An important special case of the above theorems arises as follows. Take $X = \operatorname{Spec} R$ with R a Dedekind domain in the above (for example, $R = \mathbb{Z}$). Let $F := \operatorname{Frac} R$ be the field of fractions of R. Let $Q \colon V \to F$ be a nondegenerate binary quadratic form and let $K := \mathrm{Clf}^0(Q)$ be its even Clifford algebra, a quadratic F-algebra. Then Equation (1.3.1) and Equation (1.3.2) specialize to the following statement.

Corollary 1.6.1. *Clifford and norm define mutually inverse discriminant-preserving bijections between the set of similarity classes of R-lattices in V and the set of invertible ideal classes for quadratic orders $S \subseteq K$. In particular, there is a bijection between the set of similarity classes of R-lattices $M \subset V$ with multiplicator ring S and the group* $\operatorname{Pic} S$.

This corollary bears mention first because the orientations have disappeared: this is because every R-lattice has a canonical S-orientation, obtained by restricting the tautological orientation $\mathrm{Clf}^0(Q) = K$ on V. The value line bundle has also disappeared, taken to be the fractional ideal generated by the values of the quadratic form on the lattice (so by construction, every quadratic module is primitive). And if we restrict to isometries then we obtain a bijection with the infinite group $\mathrm{Pic}^{(1)} S$, the group of fractional ideals of S modulo principal ideals of norm 1 (from K to F).

1.7. **Contribution.** The first contribution of this paper is to synthesize the work of Kneser [Kne82] and Wood [Woo11]. Kneser approaches composition laws via the even Clifford algebra and the norm without a categorical equivalence, whereas Wood provides a categorical equivalence and geometric approach without a composition law and without Clifford and the norm. Our theorems combine and recover their results. In particular, we show that the functor defined by Wood is naturally isomorphic to the norm functor composed with a duality by showing there is a unique such functor which restricts to the usual norm. For further context on other related work, see section 7.3. As a consequence, by viewing Gauss composition over an arbitrary base through the Clifford functor, we can see it as just one instance of the beautiful exceptional isomorphisms of Lie groups in low rank n [KMRT98, §15]: indeed this paper fills in a gap in the cases $n = 1$ and $n = 3$ considered by Voight [Voi16, Voi11a].

Our second contribution is to center the quartet of orthogonal groups $\mathrm{GSO}, \mathrm{GO}, \mathrm{SO}, \mathrm{O}$. This identifies Wood's *linear binary quadratic forms* under the action of $\mathrm{GL}_1 \times \mathrm{GL}_2$ as binary quadratic forms under similarity. Previous work does not address the role of the narrow class group in Gauss composition; we see it here as arising from oriented isometries.

Third, this paper reconciles notions of orientation. Our approach follows Kneser who calls it *C-type* [Kne82, section 4]; this notion also appears in Knus–Merkurjev–Rost–Tignol [KMRT98, Definition (12.40)] in the case of odd dimension over a field and in Auel [Aue11, 1.10] in the étale case. This solves the problems caused by the standard involution on the quadratic algebra necessary to get a composition law, noticed by other authors. This notion of orientation seems particularly well-suited to this task: the set of $\mathscr{O}_Y$-orientations on a binary quadratic module form a torsor under $\mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$, so for example if $Y \to X$ is étale and X is connected, then $\mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y) \simeq \mathbb{Z}/2\mathbb{Z}$. By contrast, the set of rigidifications (or equivalently the set of determinant structures) is a torsor for $\mathbb{G}_m(X) = \mathscr{O}_X(X)^\times$, and this is a bit of overengineering in resolving the ambiguity caused by the standard involution. We view the role of the even Clifford algebra (and more generally the universal Clifford center) as a refinement of the discriminant and therefore a natural invariant to isolate.

Next, we generalize Wood's notion of *traceable* to *pseudoregular*; this notion (but without the name) was also studied in the work of Voight [Voi11a], following Gross–Lucianovic [GL09] and Bhargava [Bha04b]. We identify pseudoregularity as a general property of the Clifford functor in arbitrary rank via the formation of the *universal Clifford center*: see section 4.1.

Finally, the elaboration of the theory in the case of lattices over a domain makes it especially well-suited for our intended application: to understand (and compute with) spaces of orthogonal modular forms as algebraic modular forms as well as their relationship to Hecke quasicharacters (and then Hilbert modular forms), see section 7.2.

1.8. **Contents.** In section 2, we give terminology and notation for quadratic modules, universal framed objects, Clifford algebras. In section 3, we explore pseudoregular modules and show functorial properties of the Clfford map. In section 4, we discuss discriminants, orientations via the universal Clifford center, and rigidifications (and compare to determinant structures). In section 5, we define a norm functor and compare to Wood's result. In section 6, we prove the main theorem and derive its corollaries. Finally, in section 7, we discuss the application to orthogonal modular forms, derive the connections to genera and class sets, and review recent work generalizing Gauss composition.

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2. Background

We begin with a bit of background and setup. For further reading, see Voight [Voi16], Bischel–Knus [BK94], and Auel [Aue15].

2.1. **Quadratic modules and algebras.** Let X be a scheme. A *quadratic $\mathscr{O}_X$-module* is a triple $(\mathscr{M}, \mathscr{L}, Q)$ where $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module of finite rank, $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module, and $Q\colon \mathscr{M} \to \mathscr{L}$ is a *quadratic map*: for all open sets $U \subseteq X$, we have

(i) $Q(rx) = r^2 Q(x)$ for all $r \in \mathscr{O}_X(U)$ and $x \in \mathscr{M}(U)$; and
(ii) The map $T\colon \mathscr{M}(U) \times \mathscr{M}(U) \to \mathscr{L}(U)$ defined by $T(x,y) = Q(x+y) - Q(x) - Q(y)$ for $x, y \in \mathscr{M}(U)$ is $\mathscr{O}_X(U)$-bilinear.

We call $T\colon \mathscr{M} \times \mathscr{M} \to \mathscr{L}$ the *associated bilinear form*; we often just write $Q\colon \mathscr{M} \to \mathscr{L}$ for the triple $(\mathscr{M}, \mathscr{L}, Q)$ and refer to it as a quadratic module.

By $\operatorname{Sym}^k \mathscr{M}$ we mean the usual quotient of $\mathscr{M}^{\otimes k}$ and by $\operatorname{Sym}_k \mathscr{M}$ we mean the usual submodule. By definition, a quadratic $\mathscr{O}_X$-module is a global section of the sheaf

$$(\operatorname{Sym}_2 \mathscr{M})^\vee \otimes \mathscr{L} \cong \operatorname{Sym}^2(\mathscr{M}^\vee) \otimes \mathscr{L}. \tag{2.1.1}$$

Concisely put, the isomorphism is defined by pullback under the natural projection $\operatorname{Sym}_2 \mathscr{M} \to \operatorname{Sym}^2 \mathscr{M}$.

A *similarity* between two quadratic modules $Q\colon \mathscr{M} \to \mathscr{L}$ and $Q'\colon \mathscr{M}' \to \mathscr{L}'$ is a pair of $\mathscr{O}_X$-module isomorphisms $\varphi\colon \mathscr{M} \to \mathscr{M}'$ and $\lambda\colon \mathscr{L} \to \mathscr{L}'$ such that $Q' \circ \varphi = \lambda \circ Q$, i.e., the diagram

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{Q} & \mathscr{L} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \mathscr{M}' & \xrightarrow{Q'} & \mathscr{L}' \end{array} \tag{2.1.2}$$

commutes. A *similitude* is a similarity with $\mathscr{L} = \mathscr{L}'$, and an *isometry* is a similitude with $\lambda = \mathrm{id}$, i.e.,

$$\begin{array}{ccc} \mathscr{M} & \overset{Q}{\searrow} & \\ {\scriptstyle\varphi}\downarrow & & \mathscr{L} \\ \mathscr{M}' & \underset{Q'}{\nearrow} & \end{array} \tag{2.1.3}$$

When $Q = Q'$, then the self-similitudes and self-isometries form groups (indeed group schemes).

A *quadratic $\mathscr{O}_X$-algebra* is an $\mathscr{O}_X$-algebra (with 1) that is locally free of rank 2 as an $\mathscr{O}_X$-module. Equivalently, a quadratic $\mathscr{O}_X$-algebra is given by $\mathscr{O}_Y$ where $Y \to X$ is a locally free morphism of degree 2. Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra. Then there is a unique *standard involution* σ on $\mathscr{O}_Y$. The *discriminant* of $\mathscr{O}_Y$ is the quadratic map $d\colon (\bigwedge^2 \mathscr{O}_Y)^{\otimes 2} \to \mathscr{O}_X$ defined by $d((x \wedge y) \otimes (z \wedge w)) = (x\sigma(y) - \sigma(x)y)(z\sigma(w) - \sigma(z)w)$ for all $x, y, z, w \in \mathscr{O}_Y(U)$ with $U \subseteq X$ open [Voi16, Proposition 3.13].

Since the objects (e.g., quadratic modules or algebras) we work with are usually locally free, we usually choose trivializations so that we can work in coordinates. We will often consider universal framed objects over the universal coordinate ring R which represents the moduli of framed data.

A *framed $\mathscr{O}_X$-module* of rank m over $\mathscr{O}_X$ is a tuple $(\mathscr{M}, \beta)$ where $\mathscr{M} = \mathscr{O}_X e_1 \oplus \cdots \oplus \mathscr{O}_X e_m$ is a free $\mathscr{O}_X$-module with (ordered) basis $\beta = \{e_1, \ldots, e_m\}$. A *framed quadratic $\mathscr{O}_X$-module* is a quadratic module $Q\colon \mathscr{M} \to \mathscr{L}$ where $\mathscr{M}$ and $\mathscr{L}$ are framed (of ranks m and 1, respectively). An isomorphism of framed quadratic $\mathscr{O}_X$-modules is a similarity which preserves the framing. The automorphism group of a framed quadratic $\mathscr{O}_X$-module is therefore trivial.

We define a functor $\mathcal{Q}^{\mathrm{fr}}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ which assigns to each scheme X the set of isomorphism classes of framed quadratic $\mathscr{O}_X$-modules of rank m under isometries preserving the framing and assigns to a morphism of schemes $f\colon X' \to X$ the pullback along f.

Lemma 2.1.4. *Let $R := \mathbb{Z}[a_{ij}]_{1\le i\le j\le m}$ be a polynomial ring in $m(m+1)/2$ variables over $\mathbb{Z}$. Then functor $\mathcal{Q}^{\mathrm{fr}}$ is represented by $Z = \operatorname{Spec}(R)$, with universal object is*

$$\begin{aligned} Q\colon \mathscr{O}_Z^m &\to \mathscr{O}_Z \\ Q(x_1,\dots,x_m) &= \sum_{1\le i\le j\le m} a_{ij}x_ix_j. \end{aligned} \tag{2.1.5}$$

Proof. Over any scheme X, a framed quadratic form $Q\colon \mathscr{M} \to \mathscr{L}$ with $\beta = e_1,\dots,e_m$ an $\mathscr{O}_X$-basis for $\mathscr{M}$ and $\mathscr{L} = \mathscr{O}_X g$ is uniquely determined by the values $a_{ii} = Q(e_i)g$ for $i = 1,\dots,m$ and $a_{ij} = (Q(e_i+e_j) - Q(e_i) - Q(e_j))g$ with $1 \le i \le j \le m$. □

In particular, framed binary quadratic modules are parametrized by $\mathbb{A}^3_{\mathbb{Z}}$ in the familiar way.

2.2. **Clifford map.** For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ (equivalently, locally free $\mathscr{O}_X$-module of rank 1), we define $\mathscr{L}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$.

Let $Q\colon \mathscr{M} \to \mathscr{L}$ be a quadratic $\mathscr{O}_X$-module. We define the *even tensor $\mathscr{O}_X$-algebra* by

$$\mathscr{T}en^0(\mathscr{M};\mathscr{L}) := \bigoplus_{d=0}^{\infty} (\mathscr{M}\otimes\mathscr{M}\otimes\mathscr{L}^\vee)^{\otimes d}. \tag{2.2.1}$$

Let $\mathscr{J}^0(Q)$ be the two-sided ideal of $\mathscr{T}en^0(\mathscr{M};\mathscr{L})$ defined over $U \subseteq X$ open by

$$\begin{aligned} &\langle x\otimes x\otimes f - f(Q(x)) : x\in\mathscr{M}(U), f\in\mathscr{L}^\vee(U)\rangle \\ &\quad + \langle x\otimes y\otimes y\otimes z\otimes f\otimes g - f(Q(y))\, x\otimes z\otimes g : x,y,z\in\mathscr{M}(U), f,g\in\mathscr{L}^\vee(U)\rangle. \end{aligned} \tag{2.2.2}$$

We define the *even Clifford algebra* of Q to be the quotient

$$\mathrm{Clf}^0(Q) := \mathscr{T}en^0(\mathscr{M};\mathscr{L})/\,\mathscr{J}^0(Q). \tag{2.2.3}$$

Proposition 2.2.4. *The even Clifford map defines a functor between*

quadratic $\mathscr{O}_X$-module under similarities

and

$\mathscr{O}_X$-algebras under $\mathscr{O}_X$-algebra isomorphisms.

Proof. See Auel [Aue15, Proposition 1.2] and for a slightly different but equivalent construction using the Rees algebra, see Bichsel–Knus [BK94, §3]. □

Example 2.2.5. In the binary case, we have a canonical isomorphism

$$\mathrm{Clf}^0(Q)/\mathscr{O}_X \cong \textstyle\bigwedge^2\mathscr{M}\otimes\mathscr{L}^\vee$$

as $\mathscr{O}_X$-modules.

If $X = \operatorname{Spec} R$ is affine with $S = \mathrm{Clf}^0(Q)$, then we get a noncanonical splitting

$$S \simeq R\oplus S/R \simeq R\oplus(\textstyle\bigwedge^2 M\otimes L^\vee).$$

See Voight [Voi16, Lemma 3.2, Remark 3.3] for further freeness comments.

Let

$$\mathscr{T}en^1(\mathscr{M};\mathscr{L}) := \bigoplus_{\substack{d=1\\ d\text{ odd}}}^{\infty} \mathscr{M}^{\otimes d}\otimes(\mathscr{L}^\vee)^{\otimes\lfloor d/2\rfloor}. \tag{2.2.6}$$

Then $\mathscr{T}en^1(\mathscr{M};\mathscr{L})$ is a graded $\mathscr{T}en^0(\mathscr{M};\mathscr{L})$-bimodule under the tensor multiplication law. Let $\mathscr{J}^1(Q)$ be the $\mathscr{T}en^0(\mathscr{M};\mathscr{L})$-sub-bimodule of $\mathscr{T}en^1(\mathscr{M};\mathscr{L})$ defined over $U \subseteq X$ open by

$$\langle x \otimes x \otimes y \otimes f - f(Q(x))y, y \otimes x \otimes x \otimes f - f(Q(x))y : x, y \in \mathscr{M}(U), f \in \mathscr{L}^\vee(U)\rangle. \quad (2.2.7)$$

We define the *odd Clifford bimodule* of Q to be the quotient

$$\mathrm{Clf}^1(Q) := \mathscr{T}en^1(\mathscr{M};\mathscr{L})/\mathscr{J}^1(Q). \quad (2.2.8)$$

Then under the natural multiplication, $\mathrm{Clf}^1(Q)$ has the structure of a $\mathrm{Clf}^0(Q)$-bimodule. Its formation is also functorial, but we first characterize a key property in the next section.

3. Pseudoregularity

In this section, we explore modules which from the point of view of characteristic polynomials look like the regular module (affording the regular representation). We then show this holds for the odd Clifford bimodule over the even Clifford algebra.

3.1. **Pseudoregular modules.** We first work locally. Let R be a commutative ring (with 1). Let A be an R-algebra (with 1, not necessarily commutative) that is free of rank n as an R-module. Left multiplication defines an R-algebra homomorphism $\lambda_A \colon A \to \mathrm{End}_R(A) \simeq \mathrm{M}_n(R)$, called the *(left) regular representation* of A (over R). We obtain for each $\alpha \in A$ a (left) characteristic polynomial $c_A(\alpha;T) = \det(T - \lambda_A(\alpha)) \in R[T]$.

Now let I be a left A-module. We say that I is *R-rank balanced* if I is also free of rank n as an R-module.

Suppose that I is R-rank balanced. Then again the left A-module structure on I yields an R-algebra homomorphism $\lambda_I \colon A \to \mathrm{End}_R(I) \simeq \mathrm{M}_n(R)$ and $c_I(\alpha;T) \in R[T]$.

Definition 3.1.1. We say I is *(R,A)-pseudoregular* if I is R-rank balanced and for all $\alpha \in A$ we have $c_A(\alpha;T) = c_I(\alpha;T)$.

The name is justified as I looks like the regular representation of A from the point of view of the numerical data encoded in characteristic polynomials over R. When clear from context, we will often drop the prefix (R,A) and say simply that I is pseudoregular.

Remark 3.1.2. One could extend Equation (3.1.1) to the case where M is a left A-module that is free R-module of rank nd with $d \geq 1$ and ask that $c_M(x;T) = c_A(x;T)^d$.

Lemma 3.1.3. *The following statements hold.*

(a) *If I is free as an A-module, then I is pseudoregular.*
(b) *If $R \to R'$ is a flat ring homomorphism and $I \otimes_R R'$ is $(R', A \otimes_R R')$-pseudoregular, then I is (R,A)-pseudoregular.*
(c) *If A is a (commutative) domain, then I is pseudoregular.*

Proof. For part (a), writing $I = Ae$, an R-basis β for A yields βe an R-basis for I, and accordingly $[\lambda_A(\alpha)]_\beta = [\lambda_I(\alpha)]_{\beta e}$ so in particular $c_A(\alpha;T) = c_I(\alpha;T)$.

For (b), let $I' := I \otimes_R R'$ and $A' := A \otimes_R R'$. For $\alpha \in A$, since $R \to R'$ is flat, then

$$\begin{aligned} c_{I'}(\alpha;T) &= \det(T - \lambda_{I'}(\alpha)) = \det(T - \lambda_{I'}(\alpha) \otimes_R 1) \\ &= \det(T - \lambda_{I'}(\alpha)) \otimes_R 1 = c_I(x;T) \otimes_R 1 \in R'[T]. \end{aligned} \quad (3.1.4)$$

Similarly, we have $c_{A'}(\alpha;T) = c_A(\alpha;T) \otimes_R 1$. By pseudoregularity we have $c_{I'}(\alpha;T) = c_{A'}(\alpha;T)$, and we conclude that I is (R,A)-pseudoregular by flatness.

For (c), apply (b) with the map $R \to F$, we see that $A \otimes_R F$ is a field so $I \otimes_R F$ is free. □

Example 3.1.5. Let $A := R \times R$, let $\mathfrak{p} = R \times \{0\}$ and let $I := A/\mathfrak{p} \oplus A/\mathfrak{p} \simeq R \oplus R$. Then $\operatorname{rank}_R A = \operatorname{rank}_R I = 2$, but I is not pseudoregular because $\alpha = (a,b)$ has $c_A(\alpha;T) = (T-a)(T-b)$, but $c_I(\alpha;T) = (T-b)^2$. Indeed, I is not faithful as an A-module, as $(1,0)$ annihilates I.

Example 3.1.6. Let $A = \mathrm{M}_2(R)$ and $I = R^2 \oplus R^2$, with action $\alpha \cdot (v,w) = (\alpha v, 0)$. Then $c_A(\alpha;T) = \det(T-\alpha)^2$ but $c_I(\alpha;T) = \det(T-\alpha)T^2$, so I is not pseudoregular and I is faithful as an A-module.

We now define the global notion. First, if A is a *locally* free R-algebra of rank n and I is an S-module which is locally free of rank n as an R-module, then we again have (uniquely defined) characteristic polynomials, so we just repeat Equation (3.1.1).

Finally, let X be a scheme, let $\mathscr{A}$ be a locally free $\mathscr{O}_X$-algebra. Let $\mathscr{I}$ be an $\mathscr{A}$-module which is locally free of rank n as a $\mathscr{O}_X$-module.

Definition 3.1.7. We say $\mathscr{I}$ is $(\mathscr{O}_X, \mathscr{A})$-*pseudoregular* if there exists an affine open cover $X = \bigcup_i U_i$ such that $\mathscr{I}(U_i)$ is $(\mathscr{O}_X(U_i), \mathscr{A}(U_i))$-pseudoregular for all i.

It is straightforward to check that $\mathscr{I}$ is pseudoregular if and only if for all affine open $U \subseteq X$ we have $\mathscr{I}(U)$ is pseudoregular.

We make similar definitions for right modules and for bimodules (pseudoregular as both a left and right module).

Proposition 3.1.8. $\mathrm{Clf}^1(Q)$ *is pseudoregular as a* $\mathrm{Clf}^0(Q)$*-bimodule.*

Proof. We first work universally, using the universal framed quadratic module Q (2.1.5). Let $S := \mathrm{Clf}^0(Q)$ and $I := \mathrm{Clf}^1(Q)$ be the resulting (universal) framed even Clifford algebra and odd Clifford bimodule. Let $\xi := \sum_{1 \le i \le j \le n} z_{ij} e_i e_j \in R' := R[z_{ij}]_{i,j}$ be the universal element of S. Let $S' := S \otimes_R R'$ and similarly I'. We claim that I' is (R', S')-pseudoregular. Indeed, since R' is a domain, by Equation (3.1.3)(b) it is enough to check this over $F' := \operatorname{Frac} R'$. But then I' is free over S' with basis e_1 (essentially since $e_1^2 = a_{11} \in (F')^\times$), so the claim follows from Equation (3.1.3)(a). In particular, we get that

$$c_{S'}(\xi;T) = c_{I'}(\xi;T) \in R'[T]. \tag{3.1.9}$$

Now for the general case. As the notion is local, we reduce to the case of an affine cover where the objects are free. So let $Q\colon M \to L$ be a quadratic module with $M = R^n$ and $L = R$. Let $S := \mathrm{Clf}^0(Q)$ and $I := \mathrm{Clf}^1(Q)$, and let $\alpha \in S$; we want to show that $c_S(\alpha;T) = c_I(\alpha;T)$. But Q and α are specializations of the universal quadratic form and ξ under a unique ring homomorphism by Equation (2.1.4), so the result follows by specializing (3.1.9). □

Proposition 3.1.10. *The association of the odd Clifford bimodule over the even Clifford algebra defines a functor between*

quadratic $\mathscr{O}_X$*-modules under similarities*

and

pseudoregular bimodules over $\mathscr{O}_X$*-algebras under semilinear module isomorphisms.*

Proof. See [Aue15, Proposition 1.5] together with Equation (3.1.8). □

Proposition 3.1.11. *The association* $Q \to (\mathrm{Clf}^0(Q), \mathrm{Clf}^1(Q))$ *gives a functor fibered over* **Sch** *from the category*

quadratic $\mathscr{O}_X$*-modules under similarities*

to the category

pairs $(\mathscr{A}, \mathscr{I})$ *where* $\mathscr{A}$ *is a locally free* $\mathscr{O}_X$*-algebra and* $\mathscr{I}$ *is a pseudoregular* $\mathscr{A}$*-bimodule under semilinear module isomorphisms.*

Proof. It is clear from the Clifford construction that objects go to objects in the above, with pseudoregularity implied by Equation (3.1.8). The functorial properties in [Aue11, Section 4] shows the fact that similarites associate to similinear module isomorphisms between pairs $(\mathscr{A}, \mathscr{I})$. □

3.2. **Quadratic pseudoregular modules.** We now specialize to the quadratic case, first exhibiting a universal object.

Indeed, we similarly consider framed objects for $\mathscr{O}_X$-algebras and pseudoregular modules over them. A *framed unital $\mathscr{O}_X$-algebra* is an $\mathscr{O}_X$-algebra with a framing β such that $\beta_1 = 1$.

A $\mathscr{O}_X$-rank balanced $\mathscr{O}_Y$-module $\mathscr{I}$ is *framed* if both $\mathscr{O}_Y$ and $\mathscr{I}$ are framed. An isomorphism of $\mathscr{O}_X$-rank balanced modules is an isomorphism of modules which preserves the framing: more precisely, a pair (φ, ψ) where $\varphi\colon \mathscr{O}_Y \to \mathscr{O}_{Y'}$ is an $\mathscr{O}_X$-algebra isomorphism preserving framings and $\psi\colon \mathscr{I} \to \mathscr{I}'$ a φ-semilinear module isomorphism that also preserves framings.

Let $\mathscr{I}$ be a framed quadratic $\mathscr{O}_X$-rank balanced module, with $1, \gamma$ an $\mathscr{O}_X$-basis for $\mathscr{O}_Y$ and e_1, e_2 a $\mathscr{O}_X$-basis for $\mathscr{I}$. Then $\gamma^2 - t\gamma + n = 0$ with $t, n \in \mathscr{O}_X(X)$ and the matrix of γ in the given basis is of the form $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $a, b, c, d \in \mathscr{O}_X(X)$. Following Wood [Woo11, proof of Theorem 2.1], replacing $\gamma \leftarrow \gamma - d$, we see that we can arrange $d = 0$. But then following Gross–Lucianovic [GL09] and Voight [Voi11a], we say that e_1, e_2 is a *good basis* for $\mathscr{I}$ (over $\mathscr{O}_Y$) and altogether they give a *good frame*.

Example 3.2.1. For a framed binary quadratic module with form $ax^2 + bxy + cy^2$, we have $\mathrm{Clf}^0(Q) = \mathscr{O}_X \oplus \mathscr{O}_X e_1 e_2$ and $\mathrm{Clf}^1(Q) = \mathscr{O}_X e_1 \oplus \mathscr{O}_X e_2$, and this is a good frame: we have $(e_1e_2)e_1 = e_1(b - e_1e_2) = be_1 - ae_2$ and $(e_1e_2)e_2 = ce_1$ so $[e_1e_2] = \begin{pmatrix} b & -a \\ c & 0 \end{pmatrix}$.

We define a functor $\mathcal{P}^{\mathrm{fr}}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ which assigns to each scheme X the set of isomorphism classes of good framed quadratic $\mathscr{O}_X$-rank balanced modules, and to a morphism assigns the pullback.

Lemma 3.2.2. *Let* $R := \mathbb{Z}[a, b, c, t, n]/(t - a, n + bc) \simeq \mathbb{Z}[a, b, c]$. *Then the functor* $\mathcal{P}^{\mathrm{fr}}$ *is represented by* $Z = \mathrm{Spec}(R)$, *with universal object* $\mathscr{I} = \mathscr{O}_X e_1 \oplus \mathscr{O}_X e_2$ *over* $\mathscr{O}_Y = \mathscr{O}_X \oplus \mathscr{O}_X \alpha$ *satisfying* $\alpha^2 - t\alpha + n = 0$ *and acting by* $[\alpha] = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$.

Proof. As in the setup of Wood [Woo11, Proof of Theorem 1.4], there are no automorphisms of a good framed quadratic rank-balanced quadratic module, and it is uniquely determined by the data t, n, a, b, c by linearity; the relations come from the fact that the characteristic polynomial matches, so $a = t$ and $n = -bc$. □

Indeed, in the quadratic case one only needs to check traces, as follows.

Lemma 3.2.3. *Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra and let $\mathscr{I}$ be an $\mathscr{O}_Y$-module which is locally free of rank 2 as an $\mathscr{O}_X$-module. Then $\mathscr{I}$ is pseudoregular if and only if for all open $U \subseteq X$ and all $\alpha \in \mathscr{O}_Y(U)$ we have $\mathrm{Tr}_{\mathscr{O}_Y(U)}(\alpha) = \mathrm{Tr}_{\mathscr{I}(U)}(\alpha)$.*

Therefore $\mathscr{I}$ is pseudoregular if and only if $\mathscr{I}$ is *traceable* in the language of Wood [Woo11].

Proof. We immediately reduce to the local case, so we may suppose that S, I are free over R of rank 2.

If $\mathscr{I}$ is pseudoregular, then reading the coefficient of T in $c_S(\alpha;T) = c_I(\alpha;T)$ we have $\mathrm{Tr}_S(\alpha) = \mathrm{Tr}_I(\alpha)$ for all $\alpha \in S$.

Conversely, suppose that $\mathrm{Tr}_S(\alpha) = \mathrm{Tr}_I(\alpha)$ for all $\alpha \in S$. It suffices to show that $\mathrm{Nm}_S(\alpha) = \mathrm{Nm}_I(\alpha)$ for all $\alpha \in S$. Let $1, \gamma, e_1, e_2$ be a good basis for S and I. Then as in the universal case, we have $\gamma^2 - t\gamma + n = 0$ with $t, n \in R$ and $\gamma e_1 = ae_1 + be_2$ and $\gamma e_2 = ce_1$ with $a, b, c \in R$. Since $\mathrm{Tr}_S(\gamma) = \mathrm{Tr}_I(\gamma)$, then $t = a$. Also, we have

$$tce_1 - ne_2 = (t\gamma - n)e_2 = \gamma^2 e_2 = ace_1 + bce_2 \tag{3.2.4}$$

and thus $\mathrm{Nm}_S(\gamma) = n = -bc = \mathrm{Nm}_I(\gamma)$. Finally, given $\alpha = x + y\gamma \in S$ with $x, y \in R$ and $[\alpha] = \begin{pmatrix} x + ay & by \\ cy & x \end{pmatrix}$ having $\det([\alpha]) = x^2 + axy - bcy^2$, we have

$$\mathrm{Nm}_S(\alpha) = (x + y\gamma)(x + y(t - \gamma)) = x^2 + txy + ny^2 = x^2 + axy - bcy^2 = \mathrm{Nm}_I(\alpha) \tag{3.2.5}$$

and hence $c_S(\alpha;T) = c_I(\alpha;T)$ as desired. □

4. Discriminants, orientations, and rigidifications

4.1. **Universal Clifford center.** In this section, we explain how to associate a quadratic algebra to every quadratic module in a discriminant-preserving way.

Recall the universal framed quadratic module and its even Clifford algebra in the proof of Equation (3.1.8). Given a quadratic $\mathscr{O}_X$-module $Q\colon \mathscr{M} \to \mathscr{L}$, the center of the universal even Clifford algebra specializes to the even Clifford algebra of Q, which we call the *universal Clifford center* of Q, denoted $\mathscr{Z}(Q)$. By construction, the formation of the universal Clifford center is functorial for similarities and commutes with base change.

Proposition 4.1.1. *If $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module of even rank, then $\mathscr{Z}(Q)$ is a quadratic $\mathscr{O}_X$-algebra.*

Proof. It is enough to check this for the center of the universal even Clifford algebra: for details, see [Aue15, Proposition 1.2]. □

Example 4.1.2. When Q is binary, we have $\mathscr{Z}(Q) = \mathrm{Clf}^0(Q)$.

4.2. **Discriminants.** In this section, we define the discriminant of Q intrinsically and show that it agrees with the discriminant of $\mathscr{Z}(Q)$.

Let T be the associated bilinear form of Q. On each open set $U = \operatorname{Spec} R$ with $\mathscr{M}(U) = M$ free of rank n and $\mathscr{L}(U) = L$, we define the *discriminant* $\mathrm{disc}(Q)$ of Q to be the determinant $\det(T)$ of T with respect to a basis of $M \simeq R^n$, which is a section of $L^{\otimes n} \otimes (\bigwedge^2 M)^{\otimes -2}$. The collection of local discriminants on an affine open cover glue to the *(global) discriminant* $\mathrm{disc}(Q)$ of Q.

Suppose that Q is binary. Let $M = Re_1 \oplus Re_2$ and $L = Rl$. Suppose that $Q(e_1) = al$, $Q(e_2) = cl$, and $T(e_1, e_2) = bl$. Then $\operatorname{disc}(Q) = \det(T) = (b^2 - 4ac)l^2$. our definition of discriminate agrees with Wood [Woo11, p. 1760]. Let $\operatorname{Clf}^0(Q)(U) = S = R \otimes Re_1e_2$ and σ be the standard involution on $\operatorname{Clf}^0(Q)$ and $d\colon (\bigwedge^2 \operatorname{Clf}^0(Q))^{\otimes 2} \to \mathscr{O}_X$ be the discriminant of $\operatorname{Clf}^0(Q)$. Locally on U, we have $d((1\wedge e_1e_2l^\vee)\otimes(1\wedge e_1e_2l^\vee)) = (\sigma(e_1e_2l^\vee) - e_1e_2l^\vee)^2 = b^2 - 4ac$. Hence, we can instead define the *discriminant* of Q to be the discriminant $d_{\mathscr{Z}(Q)}$ of $\mathscr{Z}(Q)$ which is a refinement of $\operatorname{disc}(Q)$.

4.3. **Orientations.** We now organize quadratic modules according to the invariant provided by the universal Clifford center, as follows. For more details of orientations, see the discussion and references in section 1.7.

Definition 4.3.1. Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra. An $\mathscr{O}_Y$-*orientation* of $Q\colon \mathscr{M} \to \mathscr{L}$ is an $\mathscr{O}_X$-algebra isomorphism $\iota\colon \mathscr{Z}(Q) \xrightarrow{\sim} \mathscr{O}_Y$. An *orientation* of Q is an $\mathscr{O}_Y$-orientation for some Y.

Let $Q'\colon \mathscr{M}' \to \mathscr{L}'$ be another quadratic $\mathscr{O}_X$-module with $\mathscr{O}_Y$-orientation $\iota'\colon \mathscr{Z}(Q') \xrightarrow{\sim} \mathscr{O}_Y$. We say that a similarity (φ, λ) between Q and Q' is $\mathscr{O}_Y$-*oriented* if the diagram

$$\begin{array}{ccc} \mathscr{Z}(Q) & \xrightarrow{\operatorname{Clf}^0(\varphi,\lambda)} & \mathscr{Z}(Q') \\ & {\scriptstyle\iota}\searrow \quad \swarrow{\scriptstyle\iota'} & \\ & \mathscr{O}_Y & \end{array} \tag{4.3.2}$$

commutes.

Lemma 4.3.3. *The set of $\mathscr{O}_Y$-orientations is a torsor under $\operatorname{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$ by postcompositions.*

Proof. Given an $\mathscr{O}_Y$-orientation $\iota\colon \mathscr{Z}(Q) \to \mathscr{O}_Y$, $u \circ \iota = \iota$ with $u \in \operatorname{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$ forces $u = \mathrm{id}$, and thus this implies the freeness. Given another $\mathscr{O}_Y$-orientation $\iota'\colon \mathscr{Z}(Q) \to \mathscr{O}_Y$, then $u \circ \iota = \iota'$ with $U = \iota' \circ \iota^{-1}$, and thus this implies the transitivity. □

Every Q has a canonical $\mathscr{Z}(Q)$-orientation given by the identity. If Q admits an $\mathscr{O}_Y$-orientation, then the set of possibly $\mathscr{O}_Y$-orientations is a torsor for $\operatorname{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$ (under postcomposition). In particular,

Example 4.3.4. If Y is an integral scheme (hence X is also integral) and $Y \to X$ is generically separable, then $\operatorname{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y) \simeq C_2$ with the nontrivial element given by the standard involution. Now let $X = \operatorname{Spec} F$ where F is a field and (V, Q) be a quadratic space over F. Consider the case when $\operatorname{Clf}^0(Q) \simeq F(\sqrt{d})$. Then an orientation of Q is a choice of generator for $F(\sqrt{d})$.

Corollary 4.3.5. *The association $Q \to (\operatorname{Clf}^0(Q), \operatorname{Clf}^1(Q))$ gives a functor fibered over* **Quad** *from the category*

$\mathscr{O}_Y$-oriented quadratic $\mathscr{O}_X$-modules under oriented similarities

to the category

pseudoregular $\mathscr{O}_Y$-modules under $\mathscr{O}_Y$-module isomorphisms.

Proof. By Equation (3.1.11), we only need to check that morphisms match. Let (φ, λ) be an $\mathscr{O}_Y$-oriented similarity between quadratic $\mathscr{O}_X$-modules $Q\colon \mathscr{M} \to \mathscr{L}$ and $Q'\colon \mathscr{M}' \to \mathscr{L}'$ with

respect to the orientations $\iota\colon \mathrm{Clf}^0(Q) \to \mathscr{O}_Y$ and $\iota'\colon \mathrm{Clf}^0(Q') \to \mathscr{O}_Y$. Then we define the $\mathscr{O}_Y$-actions on $\mathrm{Clf}^1(Q) = \mathscr{M}$ and $\mathrm{Clf}^1(Q') = \mathscr{M}'$ by $rx = \iota^{-1}(r)x$ and $rx' = \iota'^{-1}(r)x'$ for all $r \in \mathscr{O}_Y$, and $x \in \mathscr{M}$, and $x' \in \mathscr{M}'$. Then we have $\varphi(rx) = \varphi(\iota^{-1}(r)x) = \mathrm{Clf}^0(\varphi, \lambda)(\iota^{-1}(r))\varphi(x) = \iota'^{-1}(r)\varphi(x) = r\varphi(x)$. □

4.4. **Rigidifications.** The orientation in the previous section is quite different from that given by Wood [Woo11], O'Dorney [O'Do16], and Dallaporta [Dal24], as we now explain and put to a different purpose.

Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra, $\mathscr{I}$ be a $\mathscr{O}_Y$-module which is locally free as an $\mathscr{O}_X$-module of rank 2, and $\mathscr{L}, \mathscr{P}$ be invertible $\mathscr{O}_X$-modules.

Definition 4.4.1. We define the ($\mathscr{L}$-)*rigidification* of $\mathscr{I}$ to be an $\mathscr{O}_X$-module isomorphism $r\colon \mathscr{N}(\mathscr{I}) \to \mathscr{L}$.

With the duality in Section 5.3 in mind, the ($\mathscr{L}$-)rigidification agrees with the *$\mathscr{L}$-type orientation* of $(\mathscr{O}_Y, \mathscr{I})$ in [Woo11, section 5]. Given an $\mathscr{O}_Y$-moduel isomorphisms $\varphi\colon \mathscr{I} \to \mathscr{I}'$ where $\mathscr{I}, \mathscr{I}'$ are equipped with ($\mathscr{L}$-)rigidificaitons r and r' respectively, we say that φ is *rigidified* if the induced isomorphism $(\wedge^2\varphi) \otimes \mathrm{id}\colon \mathscr{N}(\mathscr{I}) \to \mathscr{N}(\mathscr{I}')$ commutes with the rigidifications, i.e., the diagram

$$\begin{array}{ccc} \mathscr{N}(\mathscr{I}) & \xrightarrow{(\wedge^2\varphi\otimes\mathrm{id})} & \mathscr{N}(\mathscr{I}') \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle r'} & \\ & \mathscr{L} & \end{array} \tag{4.4.2}$$

commutes.

Definition 4.4.3. We define a *$\mathscr{P}$-determinant structure* on a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank n to be an $\mathscr{O}_X$ module isomorphism $\omega\colon \bigwedge^n \mathscr{E} \to \mathscr{P}$.

We note that a ($\mathscr{L}$-)rigidification of $\mathscr{I}$ is equivalent to a $\mathscr{L} \otimes (\mathscr{O}_Y/\mathscr{O}_X)$-determinant structure of $\mathscr{I}$. Given an $\mathscr{O}_X$-module isomorphism $\varphi\colon \mathscr{E} \to \mathscr{E}'$ where $\mathscr{E}, \mathscr{E}'$ have $\mathscr{P}$-determinant structures ω, ω', we define $\det\varphi \in \mathscr{O}_X(X)$ by $\omega' \circ (\wedge^n\varphi) = (\det\varphi)\omega$.

Lemma 4.4.4. *The set of ($\mathscr{L}$-)rigidifications of $\mathscr{I}$ is a torsor under $\mathbb{G}_m(X) := \mathscr{O}_X(X)^\times$.*

Proof. The action of $u \in \mathbb{G}_m(X)$ on a ($\mathscr{L}$-)rigidification $r\colon \mathscr{N}(\mathscr{I}) \to \mathscr{L}$ is induced by the left multiplication m_u by u on $\mathscr{L}$, i.e., $u \cdot r = m_u \circ r$. Then $m_u \circ r = r$ forces $u = 1$, and thus this implies freeness. Given another ($\mathscr{L}$-)rigidification $r'\colon \mathscr{N}(\mathscr{I}) \to \mathscr{L}$, let $v = r' \circ r^{-1} \in \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{L}) \simeq \mathbb{G}_m(X)$ (because $\mathscr{L}$ is a line bundle). Then $v = m_t$ for a unique $t \in \mathbb{G}_m(x)$ and $r' = m_t \circ r$. This implies transitivity. □

Remark 4.4.5. A ($\mathscr{L}$-)rigidification of $\mathscr{I}^\vee$ agrees with the $\mathscr{L}$-type orientation defined by Wood [Woo11, section 5]; over a Dedekind domain (where we consider a frational ideal $\mathfrak{a}$ instead of $\mathscr{L}$), this also agrees with the type $\mathfrak{a}$ orientation defined by O'Dorney [O'Do16, section 2.2]. Dallaporta [Dal24, section 3.1] defines an orientation of $\mathscr{O}_Y$ to be an $\mathscr{O}_X$-module isomorphism $\mathscr{O}_Y/\mathscr{O}_X \xrightarrow{\sim} \mathscr{L}^\vee$ which recovers the orientation in [Woo11, Theorem 5.2] by setting $\mathscr{L} = \mathscr{O}_X^\vee$. The choice of dual is to deal with discriminant and parities. We note that the $\mathscr{O}_X$-module isomorphism $\bigwedge^2\mathscr{O}_Y \simeq \mathscr{O}_Y/\mathscr{O}_X$ induces the $\mathscr{L}^\vee$-determinant structure $\bigwedge^2\mathscr{O}_Y \xrightarrow{\sim} \mathscr{L}^\vee$.

5. Norms

In this section, we define the norm functor and establish its properties, comparing it to the functor of Wood.

5.1. **Norm functor.** Let $\mathscr{O}_Y$ be a quadratic $\mathscr{O}_X$-algebra and let $\mathscr{I}$ be a $\mathscr{O}_Y$-module which is locally free as an $\mathscr{O}_X$-module of rank 2 (not yet pseudoregular).

Proposition 5.1.1. *There exists a unique quadratic $\mathscr{O}_X$-module*

$$E = E_{\mathscr{I}} \colon \mathscr{I} \otimes \mathscr{O}_Y/\mathscr{O}_X \to \bigwedge^2 \mathscr{I} \otimes \mathscr{O}_Y/\mathscr{O}_X$$

with the property that

$$E(x \otimes \gamma) = (\gamma x \wedge x) \otimes \gamma. \tag{5.1.2}$$

for all $x \in \mathscr{I}(U)$ and $\gamma \in \mathscr{O}_Y(U)$ with $U \subseteq X$ open.

Proof. It is enough to show this locally, gluing by uniqueness. So let $U = \operatorname{Spec} R$, $S = \mathscr{O}_Y(U)$ a free quadratic R-algebra, and $I = \mathscr{I}(U)$ an S-module free of rank 2 over R. Let $I = Re_1 \oplus Re_2$ and $S = R \oplus R\alpha$, and let $x \in I$ and $\gamma = r\alpha + s \in S$ with $r, s \in R$. Since

$$\gamma x \wedge x = (r + s\alpha)x \wedge x = (s\alpha)x \wedge x = s(\alpha x \wedge x) \tag{5.1.3}$$

the map is indeed well-defined on $I \otimes S/R$. Then

$$I \otimes S/R \simeq R(e_1 \otimes \alpha) \oplus R(e_2 \otimes \alpha) \tag{5.1.4}$$

and an arbitrary element is of the form $x \otimes \alpha$ with $x = x_1e_1 + x_2e_2$. In particular, the map is uniquely defined by (5.1.2). We similarly have

$$\bigwedge^2 I \otimes S/R = R((e_1 \wedge e_2) \otimes \alpha) \tag{5.1.5}$$

and

$$E(x \otimes \alpha) = (\alpha x \wedge x) \otimes \alpha$$

as a map of sets $E \colon I \otimes S/R \to \bigwedge^2 I \otimes S/R$.

To see it is a quadratic map, let $\beta = \{e_1, e_2\}$ and $[\alpha]_\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Identifying standard bases as usual, we have

$$\begin{aligned} \alpha x \wedge x = [\alpha]_\beta [x]_\beta \wedge [x]_\beta &= \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \wedge \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= (x_2(ax_1 + bx_2) - x_1(cx_1 + dx_2))(e_1 \wedge e_2) = E(x_1, x_2)(e_1 \wedge e_2) \end{aligned} \tag{5.1.6}$$

where

$$E(x_1, x_2) := -cx_1^2 + (a - d)x_1x_2 + bx_2^2 \tag{5.1.7}$$

is indeed a quadratic form, giving

$$E(x \otimes \alpha) = E(x_1, x_2)((e_1 \wedge e_2) \otimes \alpha) \tag{5.1.8}$$

and finishing the proof. □

We call the map E in Equation (5.1.1) the *canonical exterior form* (associated to $\mathscr{I}$, over $\mathscr{O}_Y$). To restore the domain as $\mathscr{I}$, we twist; let

$$\mathscr{N}(\mathscr{I}) := \bigwedge^2 \mathscr{I} \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee \cong (\bigwedge^2 \mathscr{I} \otimes \mathscr{O}_Y/\mathscr{O}_X) \otimes (\mathscr{O}_Y/\mathscr{O}_X)^{\vee 2}. \tag{5.1.9}$$

We define the *norm map* to be the twist by $(\mathscr{O}_Y/\mathscr{O}_X)^\vee$:

$$N_{\mathscr{I}} := E_{\mathscr{I}} \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee \colon \mathscr{I} \to \mathscr{N}(\mathscr{I}) \tag{5.1.10}$$

When we deal with similitudes, we often need to match the value line bundles for the norm maps. Given a ($\mathscr{L}$-)regidification $r\colon \mathscr{N}(\mathscr{I}) \to \mathscr{L}$, we define the *modified norm map* $N_{\mathscr{I},\mathscr{L}}$ to be the composition $N_{\mathscr{I},\mathscr{L}} = r \circ N_{\mathscr{I}}$.

Remark 5.1.11. It is interesting to note that the norm functor is defined on all $\mathscr{O}_Y$-modules $\mathscr{I}$ that are locally free $\mathscr{O}_X$-modules of rank 2, not necessarily pseudoregular. This is analogous to the role of the *exceptional rings* (which are not quaternion rings) in the case of ternary quadratic forms [GL09, Voi11a], and so this suggests a generalization in higher rank.

Lemma 5.1.12. *The norm map has*

$$N_{\mathscr{I}}(\gamma x) = \mathrm{Nm}(\gamma) N_{\mathscr{I}}(x) \tag{5.1.13}$$

for all $x \in \mathscr{I}(U)$ and $\gamma \in \mathscr{O}_Y(U)$.

Proof. The twist in N does not affect property (5.1.13), so it is equivalent to show it for $E_{\mathscr{I}}$. We can check this property locally, so we refer to (5.1.8). Again let $\gamma = r\alpha + s$ with $r, s \in R$, so that $[\gamma]_\beta = r[\alpha]_\beta + s$ and

$$[\gamma]_\beta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (ra+s)x_1 + rbx_2 \\ rcx_1 + (rd+s)x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \tag{5.1.14}$$

so that

$$E(\gamma x \otimes \alpha) = E(x_1', x_2')((e_1 \wedge e_2) \otimes \alpha)$$

and then

$$\begin{aligned} E(x_1', x_2') &= -c(x_1')^2 + (a-d)x_1'x_2' + b(x_2')^2 \\ &= ((ad-bc)r^2 + (a+d)rs + s^2)(-cx_1^2 + (a-d)x_1x_2 + bx_2^2) \\ &= \mathrm{Nm}(r\alpha + s) E(x_1, x_2) \end{aligned} \tag{5.1.15}$$

as claimed. □

Remark 5.1.16. In [Kne82], a quadratic R-module $Q\colon M \to L$ is called *of type* C if C is a quadratic R-algebra, M is a projective C-module of rank 1, and $Q(c\alpha) = \mathrm{nrd}(c)Q(\alpha)$ for all $c \in C$ and $\alpha \in M$, where nrd is the reduced norm on C. Kneser shows the following. Suppose that $\mathrm{Ann}(Q(M)) := \{r \in R\colon rQ(M) = 0\} = \{0\}$. Then there is a unique R-algebra homomorphism $\beta\colon \mathrm{Clf}^0(Q) \to C$ such that $\beta(c)\alpha = c\alpha$ for all $c \in \mathrm{Clf}^0(Q)$ and $\alpha \in M$. If Q is primitive, then β is an isomorphism. Our notion basically equips the modules with the type given by the Clifford map.

Example 5.1.17. Suppose $\mathscr{I} = \mathscr{O}_Y$ is trivial. Then $\mathscr{N}(\mathscr{I}) = \bigwedge^2 \mathscr{I} \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee = \bigwedge^2 \mathscr{O}_Y \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee \cong \mathscr{O}_X$, the latter under the map $(\alpha \wedge \gamma) \otimes f \mapsto f(\alpha\gamma)$. Equation (5.1.12) then implies that $N_{\mathscr{I}}$ is just the usual norm $\mathrm{Nm}\colon \mathscr{O}_Y \to \mathscr{O}_X$.

Theorem 5.1.18. *The association $\mathscr{I} \mapsto N_{\mathscr{I}}$ gives a functor from the category of*

$\mathscr{O}_Y$-modules under semilinear module isomorphisms

to the category of

quadratic $\mathscr{O}_X$-modules under similarities.

This association is the unique *functor from the category of free $\mathscr{O}_Y$-modules with the property that if $\mathscr{I} = \mathscr{O}_Y$, then $\mathscr{I} \mapsto (\mathrm{Nm}_{\mathscr{O}_Y} \colon \mathscr{O}_Y \to \mathscr{O}_X)$ maps to the usual norm.*

Proof. In Equation (5.1.1), we see that objects correspond to objects. To check morphisms, let $\psi \colon \mathscr{I} \to \mathscr{I}'$ be a semilinear $\mathscr{O}_Y$-module isomorphism with respect to $\sigma \in \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$; we need to check that the diagram

$$\begin{array}{ccc} \mathscr{I} & \xrightarrow{N_{\mathscr{I}}} & \mathscr{N}(\mathscr{I}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \wedge^2\psi\otimes\sigma} \\ \mathscr{I}' & \xrightarrow{N_{\mathscr{I}'}} & \mathscr{N}(\mathscr{I}') \end{array} \tag{5.1.19}$$

commutes so that $(\psi, \wedge^2\psi \otimes \sigma)$ defines a similarity. It is enough to check this for E instead, to do so locally, and confirm

$$(\wedge^2\psi\otimes\sigma)(E(x\otimes\alpha)) = (\psi(\alpha x)\wedge\psi(x))\otimes\sigma(\alpha) = (\sigma(\alpha)\psi(x)\wedge\psi(x))\otimes\sigma(\alpha) = E'(\psi(x)\otimes\sigma(\alpha)) \tag{5.1.20}$$

as in the proof of Equation (5.1.1).

Our norm functor has the desired property by Equation (5.1.17). To show uniqueness, we again work with the universal framed object. Take $R = \mathbb{Z}[a,b,c,d,t,n]/(a+d-t, ad-bc-n)$ and $S = R \oplus R\gamma$ where $\gamma^2 - t\gamma + n = 0$. Let $I = Re_1 + Re_2$ be an S-module with $\beta = \{e_1, e_2\}$ and $[\gamma]_\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(R)$. Then we obtain the norm map N_I which is uniquely determined in Equation (5.1.1). Under a unique ring homomorphism, $N_{\mathscr{I}}$ is a specialization of N_I by Equation (3.2.2). □

Corollary 5.1.21. *The association in Equation* (5.1.18) *is the* unique *functor from the category of pseudoregular $\mathscr{O}_Y$-modules with the property that $\mathscr{I} = \mathscr{O}_Y$ is associated to the usual norm map* $\mathrm{Nm}_{\mathscr{O}_Y} \colon \mathscr{O}_Y \to \mathscr{O}_X$.

Proof. Since I is pseudoregular, we can reduce to the free case by tensoring with $F = \mathbb{Q}(t,n)$. In particular, $I \otimes F$ is a free module of rank 2 over $S \otimes F$. Then we can conclude by Equation (5.1.18). □

5.2. **Involutions.** In this section, we discuss two involutions on the category of pseudoregular $\mathscr{O}_Y$-modules.

We begin with the dual. Let $\mathscr{I}$ be an $\mathscr{O}_Y$-module which is locally free of rank 2 as an $\mathscr{O}_X$-module. Then $\mathscr{I}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$ is also locally free of rank 2 as an $\mathscr{O}_X$-module.

Lemma 5.2.1. *Suppose $\mathscr{I}$ is pseudoregular. Then $\mathscr{I}^\vee$ is pseudoregular.*

Proof. Since pseudoregularity is a local property, we may work locally on an affine open subset. Let R be a commutative ring, $S = R \oplus R\alpha$ be a quadratic R-algebra, and $I = Re_1 \oplus Re_2$ be a pseudoregular S-module. Then $I^\vee = \mathrm{Hom}_R(I,R) = Re_1^* \oplus Re_2^*$ where $e_j^*(e_j) = \delta_{ij}$. Let $\lambda_I \colon S \to \mathrm{End}(I) \simeq M_n(R)$ and $\lambda_{I^\vee} \colon S \to \mathrm{End}_R(I^\vee) \simeq M_n(R)$. For each $x \in S$, we have $\lambda_I(x) = \lambda_{I^\vee}^\intercal$ and thus $c_A(x;T) = c_I(x;T) = c_{I^\vee}(x;T)$. This implies that $I^\vee$ is pseudoregular. □

Proposition 5.2.2. *The map $\mathscr{I} \mapsto \mathscr{I}^\vee$ is an contravariant autoequivalence of the category of pseudoregular $\mathscr{O}_Y$-modules.*

Proof. By Equation (5.2.1), objects are mapped to objects. A given morphism $f\colon \mathscr{I} \to \mathscr{J}$ of $\mathscr{O}_Y$-modules is mapped to the dual morphism $f^\vee\colon \mathscr{J}^\vee \to \mathscr{I}^\vee$ given by $f^\vee(\phi) = \phi \circ f$. There is a canonical evaluation $\mathscr{O}_X$ module isomorphism $\mathrm{ev}\colon \mathscr{I} \to \mathscr{I}^{\vee\vee}$ given by $\mathrm{ev}(x) = (\phi \mapsto \phi(x))$. Hence, the map is a contravariant equivalence. □

Remark 5.2.3. The dual functor is contravariant, so it reverses the direction of the morphisms. However, every morphism (e.g. $\mathscr{O}_Y$-module isomorphism or similarity) we use is an isomorphism, and thus this yields compatible morphisms under duality.

Remark 5.2.4. Fix $X = \operatorname{Spec} \mathbb{Z}$ and $Y = \operatorname{Spec} S$ where S is a quadratic order. Let $Q(x, y) = ax^2 + bxy + cy^2$ be a S-oriented primitive binary quadratic form over $\mathbb{Z}$. There are two natural involutions. Postcomposing the orientation with the standard involution on S sends Q to the binary form $ax^2 - bxy + cy^2$. This is the μ_2-action on orientations. Second, multiplying Q by the similitude factor -1 yields $-ax^2 - bxy - cy^2$. However, if we restrict to the subgroup $\mathbb{Z}^\times_{>0} = \{1\}$ as in Equation (6.2.4), this operation is no longer allowed.

5.3. **Comparison to Wood.** Wood defines a *linear binary quadratic form* as a global section $f \in \operatorname{Sym}^2 M \otimes L$, and Proposition 6.1 shows that f yields a corresponding quadratic $\mathscr{O}_X$-module $Q\colon \mathscr{M}^\vee \to L$. The dual ensures to preserve the discriminant between f and its corresponding quadratic algebra (see [Woo11, Theorem 1.6]). We therefore anticipate, in comparing to Wood, we will need to replace $\mathscr{M}$ with $\mathscr{M}^\vee$, see (2.1.1).

Let $\mathscr{L} := (\bigwedge^2 \mathscr{I})^\vee \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee$. Wood [Woo11, (1)] defines the map

$$\begin{aligned} \Psi\colon \mathscr{L}^\vee &\to \operatorname{Sym}^2 \mathscr{I} \\ (x \wedge y) \otimes \gamma &\mapsto \gamma y \otimes x - \gamma x \otimes y \end{aligned} \tag{5.3.1}$$

As in section Section 2.1, we translate this map into our (more classical) notion of binary quadratic modules. Taking the dual yields a map $\Psi^\vee\colon (\operatorname{Sym}^2 \mathscr{I})^\vee \to \mathscr{L}^{\vee\vee}$ given by $\Psi^\vee(f)(\theta) = f(\Psi(\theta))$ for all $f \in (\operatorname{Sym}^2 \mathscr{I})^\vee$ and $\theta \in \mathscr{L}^\vee$. The canonical isomorphism $(\operatorname{Sym}_2(\mathscr{I}^\vee))^\vee \cong (\operatorname{Sym}^2 \mathscr{I})^\vee$ yields a binary quadratic $\mathscr{O}_X$-module $Q_{\mathscr{I}^\vee}\colon \mathscr{I}^\vee \to \mathscr{L}$. The map $\mathscr{I} \mapsto Q_{\mathscr{I}^\vee}$ defines a contravariant functor which maps a $\mathscr{O}_Y$-module isomorphism $\psi\colon \mathscr{I} \to \mathscr{I}'$ to the similarity $(\psi^\vee, (\wedge^2\psi)^\vee \otimes \mathrm{id})$ from $Q_{\mathscr{I}'^\vee}$ to $Q_{\mathscr{I}^\vee}$.

Remark 5.3.2. Equation (5.3.1) allows us to directly obtain the corresponding binary quadratic module without passing through induced linear binary quadratic forms via [Woo11, Proposition 6.1].

Lemma 5.3.3. *The composition of the dual functor and the norm functor is naturally isomorphic to the functor $\mathscr{I} \to Q_{\mathscr{I}^\vee}$, both functors being contravariant.*

Proof. Let $\xi_{\mathscr{I}}\colon \bigwedge^2 \mathscr{I}^\vee \to (\bigwedge^2 \mathscr{I})^\vee$ be the $\mathscr{O}_X$-module isomorphism defined by

$$\xi_{\mathscr{I}}(\theta \wedge \eta) = u \wedge v \mapsto \theta(u)\eta(v) - \theta(v)\eta(u). \tag{5.3.4}$$

We claim that $(\mathrm{id}, \xi \otimes \mathrm{id})$ is a similarity from $N_{\mathscr{I}^\vee}$ to $Q_{\mathscr{I}}$, i.e. the diagram

$$\begin{array}{ccc} \mathscr{I}^\vee & \xrightarrow{N_{\mathscr{I}^\vee}} & \bigwedge^2 \mathscr{I}^\vee \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \xi_{\mathscr{I}} \otimes \mathrm{id}} \\ \mathscr{I}^\vee & \xrightarrow{Q} & (\bigwedge^2 \mathscr{I})^\vee \otimes (\mathscr{O}_Y/\mathscr{O}_x)^\vee \end{array}$$

commutes. It is enough to show this locally. Let $U = \operatorname{Spec} R$, $S = \mathscr{O}_Y(U) = R \oplus R\alpha$ a free quadratic R-algera, $I = \mathscr{I}(U) = Re_1 \oplus Re_2$ an S-module free of rank 2 over R, and $L = (\bigwedge^2 I)^\vee \otimes (S/R)^\vee$. Let $\beta = \{e_1^\vee, e_2^\vee\}$. Then we have $[\alpha]_\beta = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and thus $N_{I^\vee}(x,y) = (-bx^2 + (a-d)xy + cy^2)(e_1^\vee \wedge e_2^\vee) \otimes \alpha^\vee$ by eq. (5.1.7). Applying Equation (5.3.1), we get the map $\Psi|_U \colon L^\vee \to \operatorname{Sym}^2 I$ given by $\Psi|_U((e_1 \wedge e_2) \otimes \alpha) = \alpha e_2 \otimes e_1 - \alpha e_1 \otimes e_2 = be_1 \otimes e_1 + (d-a)e_1 \otimes e_2 - ce_2 \otimes e_2$. This yields the map $\Psi^\vee|_U \colon (\operatorname{Sym}^2 \mathscr{I})^\vee \to L$ given $\Psi^\vee|_U((e_1 \otimes e_2)^\vee) = b(e_1 \wedge e_2)^\vee \otimes \alpha^\vee$, $\Psi^\vee|_U((e_1 \otimes e_2 + e_2 \otimes e_1)^\vee) = (d-a)(e_1 \wedge e_2)^\vee \otimes \alpha^\vee$, and $\Psi^\vee|_U((e_2 \otimes e_2)^\vee) = -c(e_1 \wedge e_2)^\vee \otimes \alpha^\vee$. From the isomorphism $(\operatorname{Sym}_2(I^\vee))^\vee \cong (\operatorname{Sym}^2 I)^\vee$. we obatin the quadratic module $Q \colon I^\vee \to L$ given by $Q(xe_1^\vee + ye_2^\vee) = (bx^2 + (a-d)xy + cy^2)(e_1 \wedge e_2)^\vee \otimes \alpha^\vee$. Then we have $(\xi_{\mathscr{I}}|_U \otimes \mathrm{id})(N_{I^\vee}(x,y) = Q(x,y)$.

Next, let $\mathscr{I}'$ be another $\mathscr{O}_Y$-module which is locally free of rank 2 as an $\mathscr{O}_X$-module. Let $\psi \colon \mathscr{I} \to \mathscr{I}'$ be an $\mathscr{O}_Y$-module isomorphism. By eq. (5.1.18), the pair $(\psi^\vee, \wedge^2\psi^\vee \otimes \mathrm{id})$ defines a similarity from $N_{\mathscr{I}'^\vee}$ to $N_{\mathscr{I}^\vee}$. We want to show that the diagram

$$\begin{array}{ccc} N_{\mathscr{I}'^\vee} & \xrightarrow{(\mathrm{id}, \xi_{\mathscr{I}'} \otimes \mathrm{id})} & Q_{\mathscr{I}'} \\ \downarrow{\scriptstyle (\psi^\vee, \wedge^2\psi^\vee \otimes \mathrm{id})} & & \downarrow{\scriptstyle (\psi^\vee, (\wedge^2\psi)^\vee \otimes \mathrm{id})} \\ N_{\mathscr{I}^\vee} & \xrightarrow[(\mathrm{id}, \xi_{\mathscr{I}} \otimes \mathrm{id})]{} & Q_{\mathscr{I}} \end{array}$$

commutes. We have $(\psi^\vee \circ \mathrm{id})(\mathscr{I}'^\vee) = \mathscr{I}^\vee = (\mathrm{id} \circ \psi^\vee)(\mathscr{I}')$ and

$$\begin{aligned} (((\wedge^2\psi)^\vee \otimes \mathrm{id}) \circ (\xi_{\mathscr{I}'} \otimes \mathrm{id}))(\mathscr{N}(\mathscr{I}'^\vee)) &= ((\wedge^2\psi)^\vee \otimes \mathrm{id})((\bigwedge^2 \mathscr{I}')^\vee \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee) \\ &= (\bigwedge^2 \mathscr{I})^\vee \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee \\ &= (\xi_{\mathscr{I}} \otimes \mathrm{id})(\bigwedge^2 \mathscr{I}^\vee \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee) \\ &= ((\wedge^2\psi^\vee \otimes \mathrm{id}) \circ (\xi_{\mathscr{I}} \otimes \mathrm{id}))(\mathscr{N}(\mathscr{I}'^\vee)). \end{aligned} \tag{5.3.5}$$

□

Remark 5.3.6. Although the functor defined by Wood is covariant, all categories we work with are groupoids which are naturally equivalent to their opposite categories. Therefore, we could recover covariance by reversing isomorphisms.

5.4. **Canonical orientations.** We have a canonical $\mathscr{O}_X$-module isomorphism

$$\mathrm{Clf}^0(N_{\mathscr{I}})/\mathscr{O}_X \cong \bigwedge^2 \mathscr{I} \otimes \mathscr{N}(\mathscr{I})^\vee = \bigwedge^2 \mathscr{I} \otimes (\bigwedge^2 \mathscr{I} \otimes (\mathscr{O}_Y/\mathscr{O}_X)^\vee)^\vee \cong \mathscr{O}_Y/\mathscr{O}_X. \tag{5.4.1}$$

Proposition 5.4.2. *Suppose that $\mathscr{I}$ is pseudoregular. Then there exists a unique $\mathscr{O}_X$-algebra isomorphism*

$$\mathrm{Clf}^0(N_{\mathscr{I}}) \xrightarrow{\sim} \mathscr{O}_Y \tag{5.4.3}$$

which lifts the canonical $\mathscr{O}_X$-module isomorphism (5.4.1).

We call the isomorphism (5.4.3) the *canonical orientation*.

Proof. We will exhibit a unique such isomorphism locally. On a open set $U = \operatorname{Spec} R$, we have $S = \mathscr{O}_Y(U)$ and $I = \mathscr{I}(U)$ with a good basis $1, \gamma, e_1, e_2$. Let $N_I \colon I \to L$ where $L = \bigwedge^2 I \otimes (S/R)^\vee$. Then $\operatorname{Clf}^0(N_I) \simeq R \oplus Re_1e_2$. In the canonical isomorphism (5.4.1), locally we have $\operatorname{Clf}^0(N_I)/R \simeq R(e_1e_2) \xrightarrow{\sim} S/R = R\gamma$ where the image of e_1e_2 is $u\gamma$ with $u \in R^\times$ and we rescale e_1 so that $u = 1$.

The map $e_1e_2 \mapsto \gamma$ in fact defines an R-algebra isomorphism. It is enough to show that e_1e_2 and γ have the same characteristic polynomial. From the above, using that I is pseudoregular, we have $\operatorname{Tr}_I(\gamma) = \operatorname{Tr}_S(\gamma) = a$ and $\operatorname{Nm}_I(\gamma) = \operatorname{Nm}_S(\gamma) = -bc$. We similarly compute using (5.1.7) that $(e_1e_2)e_1 = e_1(a - e_1e_2) = ae_1 + cd_2$ and $(e_1e_2)e_2 = be_1$ giving $[e_1e_2]_\beta = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ which indeed matches. It is clear that the isomorphism is independent of the choice of good basis. □

Corollary 5.4.4. *The association $\mathscr{I} \mapsto N_{\mathscr{I}}$ together with the canonical orientation gives a functor from the category of*

$$\mathscr{O}_Y\text{-modules under isomorphisms}$$

to the category of

$$\mathscr{O}_Y\text{-oriented quadratic } \mathscr{O}_X\text{-modules under oriented similarities}$$

which is fibered over **Quad**.

Proof. The statement follows from Equation (5.1.18) together with the functoriality of the the association of the canonical orientation (by uniqueness). The fiber structure is immediate, as both categories map an object to $\mathscr{O}_Y$. □

6. Proofs of main results

In this section, we prove our main theorem and derive corollaries. The proof proceeds in several steps. We first define a quasi-inverse functor to the Clifford functor called the norm functor; we justify the name and show it nearly agrees with the construction of Wood [Woo11, (1)]. Then we show that the composition of the two is naturally isomorphic to the identity.

6.1. Main theorems.

We now conclude the proof of the main theorems.

Proof of Equation (1.3.1). We have a Clifford functor $\mathcal{F}$ by Equation (3.1.10) and a norm functor $\mathcal{G}$ by Equation (5.4.4). The discrminant-preserving property is clear from the definition. We finish by checking that the composition of the two functors in each direction is naturally isomorphic to the identity.

For each $Q \colon \mathscr{M} \to \mathscr{L}$ be an $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$ module, we have $(\mathcal{G} \circ \mathcal{F})(Q) = N_{\operatorname{Clf}^1(Q)}$ which is $\mathscr{O}_Y$-oriented. Since $\operatorname{Clf}^0(Q)/\mathscr{O}_X \simeq \bigwedge^2 \mathscr{M} \otimes \mathscr{L}^\vee$ as $\mathscr{O}_X$-modules and $\mathscr{M} = \operatorname{Clf}^1(Q)$, then there is an O_X-module isomorphism u_Q form $\mathscr{N}(\operatorname{Clf}^1(Q)) = \mathscr{N}(\mathscr{M}) = \bigwedge^2 \mathscr{M} \otimes (\operatorname{Clf}^0(Q)/\mathscr{O}_X)^\vee \simeq \bigwedge^2 \mathscr{M} \otimes (\bigwedge^2 \mathscr{M} \otimes \mathscr{L}^\vee)^\vee$ to $\mathscr{L}$, and thus (id, u_Q) is an $\mathscr{O}_Y$-oriented similarity between $N_{\operatorname{Clf}^1(Q)}$ and Q. On the other hand, for $\mathscr{I}$ be a pseudoregular $\mathscr{O}_Y$-module,

Equation (5.4.2) implies that $(\mathcal{F}\circ\mathcal{G})(\mathscr{I}) = \mathrm{Clf}^1(\mathscr{I}) = \mathscr{I}$ which is a pseudoregular $\mathrm{Clf}^0(\mathscr{I})$-module with $\mathrm{Clf}^0(\mathscr{I}) \simeq \mathscr{O}_Y$.

Next, we check the natural condition on morphisms. Let the pair of $\mathscr{O}_X$-module isomorphisms $\varphi\colon \mathscr{M} \to \mathscr{M}'$ and $u\colon \mathscr{I} \to \mathscr{I}'$ be a similarity between quadratic modules $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules $Q\colon \mathscr{M} \to \mathscr{I}$ and $Q'\colon \mathscr{M}' \to \mathscr{I}'$. Then the diagram

$$\begin{array}{ccc} N_{\mathrm{Clf}^1(Q)} & \xrightarrow{(\mathrm{id},u_Q)} & Q \\ \downarrow{\scriptstyle (\mathcal{G}\circ\mathcal{F})(\mathrm{id},u)} & & \downarrow{\scriptstyle (\varphi,u)} \\ N_{\mathrm{Clf}^1(Q')} & \xrightarrow{(\mathrm{id},u_{Q'})} & Q' \end{array} \tag{6.1.1}$$

commutes. Similarly, given pseudoregular $\mathscr{O}_Y$-modules $\mathscr{I}$ and $\mathscr{I}'$, let $\psi\colon \mathscr{I} \to \mathscr{I}'$ be an $\mathscr{O}_Y$-module isomorphism. It is clear that $(\mathcal{F}\circ\mathcal{G})(\psi) = \psi$. □

Proof of Equation (1.4.1). The canonical isomorphism in Equation (2.2.5) and the orientations together give the rigidifications with respect to the value line bundles, and thus objects map to objects. We show that oriented similitudes correspond to module isomorphisms. Let (φ,λ) be an $\mathscr{O}_Y$-oriented similitude between quadratic $\mathscr{O}_X$-modules $(\mathscr{M},\mathscr{L},Q)$ and $(\mathscr{M}',\mathscr{L},Q')$ with respect to orientations $\iota\colon \mathscr{Z}(Q) \to \mathscr{O}_Y$ and $\iota'\colon \mathscr{Z}(Q') \to \mathscr{O}_Y$ respestively. Let $\gamma\colon \bigwedge^2\mathscr{M} \otimes (\mathrm{Clf}^0(Q)/\mathscr{O}_X)^\vee \to \mathscr{L}$ and $\gamma'\colon \bigwedge^2\mathscr{M}' \otimes (\mathrm{Clf}^0(Q')/\mathscr{O}_X)^\vee \to \mathscr{L}$ be the $\mathscr{O}_X$-module isomorphisms given by Equation (2.2.5). Then φ is an $\mathscr{O}_Y$-module isomorphism, and $r\colon \mathscr{N}(\mathscr{M}) \to \mathscr{L}$ given by $r = \gamma\circ(\mathrm{id}\otimes\iota^\vee)$ and $r'\colon \mathscr{N}(\mathscr{M}') \to \mathscr{L}$ given by $r' = \gamma'\circ(\mathrm{id}\otimes\iota'^\vee)$ are ($\mathscr{L}$-)regidifications for $\mathscr{M}$ and $\mathscr{M}'$ respectively.

Conversely, let $\mathscr{I}$ and $\mathscr{I}'$ be pseudoregular $\mathscr{O}_Y$-modules equipped with $\mathscr{L}$-rigidifications $r\colon \mathscr{N}(\mathscr{I}) \to \mathscr{L}$ and $r'\colon \mathscr{N}(\mathscr{I}') \to \mathscr{L}$, and let $\varphi\colon \mathscr{I} \to \mathscr{I}'$ be an $\mathscr{O}_Y$ isomorphism. As shown in the proof of Equation (5.1.18), $(\varphi, \bigwedge^2\varphi\otimes\mathrm{id})$ is a similarity from $N_{\mathscr{I}}$ to $N_{\mathscr{I}'}$, and thus $(\varphi, r'\circ(\bigwedge^2\varphi\otimes\mathrm{id})\circ r^{-1})$ is a similitude, i.e. the diagram

$$\begin{array}{ccccc} \mathscr{I} & \xrightarrow{N_{\mathscr{I}}} & \mathscr{N}(\mathscr{I}) & \xrightarrow{r} & \mathscr{L} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bigwedge^2\varphi\otimes\mathrm{id}} & & \downarrow{\scriptstyle r'\circ(\bigwedge^2\varphi\otimes\mathrm{id})\circ r^{-1}} \\ \mathscr{I}' & \xrightarrow{N_{\mathscr{I}'}} & \mathscr{N}(\mathscr{I}') & \xrightarrow{r'} & \mathscr{L} \end{array} \tag{6.1.2}$$

commutes. A similar proof to the one of Equation (5.4.2) could show that there exist unique $\mathscr{O}_X$-algebra isomorphisms $\mathrm{Clf}^0(r\circ N_{\mathscr{I}}) \xrightarrow{\sim} \mathscr{O}_Y$ and $\mathrm{Clf}^0(r'\circ N_{\mathscr{I}'}) \xrightarrow{\sim} \mathscr{O}_Y$, and thus $(\varphi, r'\circ(\bigwedge^2\varphi\otimes\mathrm{id})\circ r^{-1})$ is $\mathscr{O}_Y$-oriented.

For the fibered structure, we map each object in the first category to the pair $(\mathscr{O}_Y,\mathscr{L})$ where $\mathscr{L}$ is its codomain; we map each object in the second one to the pair $(\mathscr{O}_Y,\mathscr{L})$ where $\mathscr{L}$ is the chosen invertible $\mathscr{O}_X$-module for the regidification. The discriminant-preserving property and the fact that the composition of the two functors in each direction is naturally isomorphic to the identity are immediate from the proof of Equation (1.3.1). □

Proof of Equation (1.5.1). We restrict the equivalence proven in Equation (1.4.1) to the subcategories with the same objects but morphisms restricted to oriented isometries and rigidified isomorphisms, respectively, and claim that this restricts again to an equivalence. It suffices to check that the Clifford functor carries the condition $\lambda = 1$ on the similitude factor to compatibility with rigidification and vice versa. We refer to (6.1.2): the right-hand square commutes, and so $\lambda = 1$ corresponds exactly to the condition of a rigidified isomorphism, and conversely. □

6.2. **Primitivity and invertibility.** In this section, we restrict our equivalences to nondegenerate objects to see the resulting restricted equivalences; as applications, taking isomorphism classes of objects, we derive relationships to Picard groups.

Definition 6.2.1. We say a quadratic $\mathscr{O}_X$-module $(\mathscr{M}, \mathscr{L}, Q)$ is *primitive* if for each open $U \subseteq X$, the sections $Q(x)$ with $x \in \mathscr{M}(U)$ generate $\mathscr{L}(U)$ as an $\mathscr{O}_X(U)$-module.

Proposition 6.2.2. *A quadratic $\mathscr{O}_X$-module is primitive if and only if it corresponds under Clifford to an invertible $\mathscr{O}_Y$-module.*

Proof. Both direction can be proved locally and glued by uniqueness. First, suppose that a quadratic $\mathscr{O}_X$-module $(\mathscr{M}, \mathscr{L}, Q)$ is primitive. Let $U = \operatorname{Spec} R$. Then $M = \mathscr{M}(U) = \mathrm{Clf}^1(Q)(U)$ and $S = \mathrm{Clf}^0(Q)(U)$ are free quadratic R-modules, and $L = \mathscr{L}(U)$ is a free R-module of rank 1. Let $M = Re_1 \oplus Re_2$, $L = R\alpha$, and $S = R \oplus Re_1e_2$ with $Q(e_1) = a\alpha, T(e_1, e_2) = b\alpha$, and $Q(e_2) = c\alpha$. Then there exist $x, y \in R$ such that $Q(xe_1 + ye_2) \in R^\times\alpha$. Let the map $\phi\colon S \to M$ be defined by $\phi(s) = (xe_1 + ye_2)s$. Let $\beta = e_1, e_2$. Since $[\phi]_\beta = \begin{pmatrix} x & -cy \\ y & ax + by \end{pmatrix}$ which has discriminant $(ax^2 + bxy + cy^2)\alpha$, then ϕ is surjective. This implies that $M = (xe_1 + ye_2)S$, and thus globally $\mathrm{Clf}^1(Q)$ is an invertible $\mathrm{Clf}^0(Q)$-module.

Conversely, suppose that $\mathrm{Clf}^1(Q)$ is an invertible $\mathrm{Clf}^0(Q)$-module. Then there exists $x, y \in R$ such that the left multiplication map $\phi_v\colon S \to M$ is an S-module isomorphism. Since ϕ_v is R-linear of quadratic R-modules, then $[\phi]_\beta \in R^\times$ and thus $Q(v) \in R^\times\alpha$. Hence, we conclude that Q is primitive. □

Now we prove the corollaries with respect to the isomorphism classes.

Proof of Equation (1.3.2). Immediate from taking isomorphism classes on both sides of Equation (1.3.1) by Equation (6.2.2). □

Example 6.2.3. If $X = \operatorname{Spec} \mathbb{Z}$, then $Y = \operatorname{Spec} S$ is a quadratic ring with discriminant $d \in \mathbb{Z}$. Then Equation (1.3.2) says that primitive binary quadratic forms of discriminant d up to $\mathrm{SL}_2(\mathbb{Z})$-equivalence and scaling by ± 1 are in bijection with the class group of S. The orientation corresponds to a fixed choice of generator of $S/\mathbb{Z}$.

Proof of Equation (1.3.3). By Equation (3.1.10) and Equation (5.1.18), similarities correspond to semilinear module isomorphisms which are determined by $\mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_Y)$. Taking isomorphism classes together with Equation (1.3.1) yields the result. □

Example 6.2.4. If $Y = \operatorname{Spec} S$ is a real quadratic ring over $X = \operatorname{Spec} \mathbb{Z}$ with discriminant d and $H = \{1\}$, then Equation (1.5.2) recovers the classical statement of Gauss composition saying that primitive binary quadratic forms of discriminant d up to $\mathrm{SL}_2(\mathbb{Z})$-equivalence are in bijection with the narrow class group of S, and orientation corresponds to a fixed choice of generator of $S/\mathbb{Z}$.

7. Applications and discussion

We conclude in this section with some applications and further discussion.

7.1. **Genera and class sets.** In this section, we classify class sets of genera which encode information about isomorphism of objects locally at all points in X. For primitive binary forms, we will see that we only need to consider global isomorphism classes.

For a quadratic $\mathscr{O}_X$-module $(\mathscr{M}, \mathscr{L}, Q)$ and a point $x \in X$, let $(\mathscr{M}_x, \mathscr{L}_x, Q_x)$ be the base extension of Q with respect to $\mathscr{O}_{X,x}$ where $\mathscr{M}_x = \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,x}$ and $\mathscr{L}_x = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,x}$. If Q has an $\mathscr{O}_Y$-orientation, then the corresponding local orientation for Q_x is $(\mathscr{O}_Y)_x$ for $x \in X$.

Let $\mathscr{O}_Y$ be an orientation of Q and $H \leq \mathscr{O}_X(X)^\times$.

- We define the *oriented similarity genus* of Q to be the set $\mathrm{Gen}_{\mathrm{GSO}}(Q)$ of quadratic $\mathscr{O}_X$-modules $(\mathscr{M}', \mathscr{L}', Q')$ such that there is an $(\mathscr{O}_Y)_x$-oriented similarity between Q_x and Q'_x for all $x \in X$.
- We can similarly define the *oriented H-similitude genus* $\mathrm{Gen}_{\mathrm{GSO},H}(Q)$.
- Forgetting orientations, we can also define the *similarity genus* $\mathrm{Gen}_{\mathrm{GO}}(Q)$ and the *H-similitude genus* $\mathrm{Gen}_{\mathrm{GO},H}(Q)$.

We define the *oriented similarity class set* $\mathrm{Cls}_{\mathrm{GSO}}(Q)$ and *similarity class set* $\mathrm{Cls}_{\mathrm{GO}}(Q)$, as well as the *oriented H-similitude class set* $\mathrm{Cls}_{\mathrm{GSO},H}(Q)$ and the *H-similitude class set* $\mathrm{Cls}_{\mathrm{GO},H}(Q)$, to be the sets of global classes in the corresponding genera respectively.

Theorem 7.1.1. *All primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules are in the same oriented similarity genus.*

Proof. It is enough to show that for any primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-module $(\mathscr{M}, \mathscr{L}, Q)$, there is an $(\mathscr{O}_Y)_x$-oriented similarity between Q_x and the norm $\mathrm{Nm}_x \colon (\mathscr{O}_Y)_x \to \mathscr{O}_{X,x}$. Given $x \in X$, we have $(\mathscr{O}_Y)_x = \mathscr{O}_{X,x} \oplus \mathscr{O}_{X,x}\alpha$, $\mathscr{M}_x = \mathscr{O}_{X,x}e_1 \oplus \mathscr{O}_{X,x}e_2$ for a good basis $1, \alpha, e_1, e_2$ of $(\mathscr{O}_Y)_x$ and $\mathscr{M}_X$. We also have $\mathscr{L}_x = \mathscr{O}_{X,x}l$ for some $l \in \mathscr{L}_x$. By Equation (5.4.2), there is a unique $\mathscr{O}_{X,x}$-algebra isomorphism between $\mathrm{Clf}^0(Q_x)$ and $(\mathscr{O}_Y)_x$ given by the map $e_1e_2 \mapsto \alpha$. Suppose that $Q_x(xe_1 + ye_2) = (ax^2 + bxy + cy^2)l$ for $a, b, c \in \mathscr{O}_{X,x}$. Then we have $(e_1e_2)^2 = be_1e_2 - ac$, and thus $\mathrm{Nm}_x(x + y\alpha) = x^2 + bxy + acy^2$. Since Q_x is primitive and $\mathscr{O}_{X,x}$ is a local ring, then there exists some x_0, y_0 such that $ax_0^2 + bx_0y_0 + cy_0^2 \in \mathscr{O}_{X,x}^\times$, and we can replace e_1 by $x_0e_1 + y_0e_2$ and extend it to a basis for $\mathscr{M}_x$. Thus, without loss of generality, we may assume that a is a unit. Let $\varphi \colon \mathscr{M}_x \to (\mathscr{O}_Y)_x$ and $\psi \colon \mathscr{L}_x \to \mathscr{O}_{X,x}$ be the $\mathscr{O}_{X,x}$-module isomorphisms defined by $\varphi(xe_1 + ye_2) = x + a^{-1}y\alpha$ and $\psi(rl) = a^{-1}r$. Then we have $\mathrm{Nm}_x(\varphi(xe_1 + ye_2)) = x^2 + a^{-1}bxy + a^{-1}cy^2 = \psi(Q_x(xe_1 + ye_2))$, and thus (φ, ψ) is a similarity between Q_x and Nm_x. The choice of good basis and Equation (5.4.2) ensure that the similarity is oriented. □

Corollary 7.1.2. *The oriented similarity classes of primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules are exactly the elements of $\mathrm{Cls}_{\mathrm{GSO}}(Q)$ for any primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-module Q.*

Proof. Taking global classes in Equation (7.1.1) yields the result. □

Fixing the value line bundle, we have the following corollary with respect to similitude.

Corollary 7.1.3. *The oriented similitude classes of primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-modules with a fixed value line bundle $\mathscr{L}$ are exactly the elements of $\mathrm{Cls}_{\mathrm{GSO},\mathscr{O}_X(X)^\times}(Q)$ of any primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$-module with value line bundle $\mathscr{L}$.*

Proof. We slightly modify the proof of Equation (7.1.1). Given $x \in X$, there is an $\mathscr{O}_{X,x}$-isomorphism $\mu\colon \mathscr{O}_{X,x} \to \mathscr{L}_x$ given by $1 \mapsto e$. Then we have $(\mu \circ \mathrm{Nm}_x)(x + y\alpha) = (x^2 + bxy + acy^2)e$. It follows that $(\phi, \mu \circ \psi)$ is a similitude between Q_x and $(\mu \circ \mathrm{Nm}_x)$. □

Next, we consider H-similitude classes with $H < \mathscr{O}_X(X)^\times$. The approach in the proof of Equation (7.1.1) does not work anymore because of the restricted choices of similitude factors. Thus, we will not realize the H-similitude classes exactly as the elements of the H-similitude class set. Instead, we pass to the side of $\mathscr{O}_Y$-modules via Clifford.

Corollary 7.1.4. *There is a bijection between the oriented H-similitude class set of a primitive $\mathscr{O}_Y$-oriented binary quadratic $\mathscr{O}_X$ module and* $\mathrm{Pic}^{(H)} Y$.

Proof. We note that the local norm at each $x \in X$ of the transition function is in H_x. Then taking global classes yields $\mathrm{Pic}^{(H)} Y$. □

7.2. **Binary orthogonal modular forms.** In this section, we will apply the classifications of class sets in Section 7.1 to understand the space of orthogonal modular forms on binary lattices.

Let F be a totally real number field and $Q\colon V \to F$ be a totally positive definite quadratic space with $\dim_F V = r$. For a $\mathbb{Z}_F$-lattice $\Lambda \subset V$ of rank r, the restriction Q_Λ of Q to Λ yields a quadratic $\mathbb{Z}_F$-module. Let $\mathfrak{N}$ be the (signed) discriminant of Λ. One can consider the equivalences, which preserve lattices, given by the reductive groups $\mathrm{O}, \mathrm{SO}, \mathrm{GO}$, or GSO. The *genus* $\mathrm{Gen}(\Lambda)$ of Λ is the set of lattices in V which becomes equivalent to Λ at each completion. The set of global equivalence classes in the genus defines the *class set* $\mathrm{Cls}(\Lambda)$. The space of *orthogonal modular forms* for Λ (of trivial weight) is the $\mathbb{C}$-vector space of functions on $\mathrm{Cls}(\Lambda)$, namely $M(\Lambda) := \{\mathrm{Cls}(\Lambda) \to \mathbb{C}\}$.

When the rank r is small, just as in the classification of semisimple Lie groups, we obtain connections between orthogonal modular forms and classical, Hilbert, and Siegel modular forms. For $r = 3$, there is a Hecke-equivariant, functorial association to classical modular forms, first exhibited by Birch [Bir91] and then generalized by Hein [Hei16] and Hein–Tornaría–Voight [HTV25]. For $r = 4$, the association to Hilbert modular forms are made explicit by Assaf–Fretwell–Ingalls–Logan–Secord–Voight [AFILSV]. For $r = 5$ and squarefree $\mathfrak{N}$, the association to Siegel paramodular forms are made explicit by Rama–Tornaría [RT20] and Dummigan–Pacetti–Rama–Tornoría [DPRT21].

The Clifford functor has played a pivotal role in the ternary and quaternary cases. In the missing binary case, Equation (7.1.2) implies the isomorphism $\mathrm{Cls}(\Lambda) \simeq \mathrm{Pic}\, S$ where $S = \mathrm{Clf}^0(Q_\Lambda)$. Then we can rewrite the space of orthogonal modular forms $M(\Lambda)$ with respect to GSO as the $\mathbb{C}$-vector space $M(\mathrm{Pic}\, S)$ of functions on $\mathrm{Pic}\, S$.

For $\mathfrak{p} \nmid \mathfrak{N}$ with $\mathfrak{p} = \mathfrak{P}\mathfrak{P}'$, we define the *Hecke operator*

$$\begin{aligned} T_\mathfrak{P}\colon M(\mathrm{Cl}(S)) &\to M(\mathrm{Cl}(S)) \\ f &\mapsto \Big([\mathfrak{A}] \mapsto f([\mathfrak{A}\mathfrak{P}]) \Big), \end{aligned}$$

and similarly for $\mathfrak{P}'$. Such operators pairwise commute and are self-adjoint with respect to a natural inner product. Thus, there is a basis of simultaneous eigenvalues, called *eigenforms*, which are exactly the multiplicative characters on $\mathrm{Pic}\, S$.

Theorem 7.2.1. *The Clifford and norm maps induce Hecke equivariant bijections between the space of orthogonal modular forms for Λ and the space of functions on* $\operatorname{Pic} S$ *where* $S = \mathrm{Clf}^0(Q_\Lambda)$.

Proof. The neighboring relation on orthogonal modular forms gives a similarity over the localization, matching up with the neighboring relation on ideal classes over the quadratic ring. □

Remark 7.2.2. For a further connection to Hilbert modular forms via theta series, see Wu [Wu25].

7.3. **Recent work.** To conclude, we review recent work generalizing Gauss composition. The history of the composition of binary quadratic forms and the revisit of Gauss composition in 19th and early 20th centuries can be found in [Dic1923] (see also [Tow80, Introduction]).

Gauss's original approach in [Gau96] was complicated. Influenced by Legendre, Dirichlet and Dedekind took another route in [Dir99, Supplement X] (see also [Cox22, Section 3A] or [Dic1951, Section IX]). Two quadratic forms $ax^2 + bxy + cy^2$ and $a'x^2 + b'xy + c'y^2$ over $\mathbb{Z}$ are called *united* if they have the same discriminant and $\gcd(a, a', \frac{b+b'}{2}) = 1$. Dirichlet and Dedekind defined a composition for united forms and realized a group structure for the set of classes of primitive positive definite forms of the same discriminant. In 1961, following Dirchilet and Dedekind, Lubelski [Lub61] generalized the composition over Euclidean rings.

In 1968, Butts–Estes [BE1968] introduced C-domains D which are integral domains with characteristic not 2 and certain structural conditions, and developed explicit composition laws for binary quadratic forms over D via united forms. They showed that the set of classes of primitive binary quadratic forms over D with discriminants in a certain set Δ is a commutative semigroup. If the discriminant is fixed, then this recovers the classical case. Moreover, they considered the quadratic algebra $R = D[\omega]$ and gave a norm condition for an R-ideal generated by A and $b+\omega$ as D-module to be invertible where A is an invertible ideal of D and $b \in D$. The direct compounds of binary quadratic forms correspond exactly to products of free rank-two R-modules, and whenever products of free D modules in R remain free, the classes of primitive form with discriminant $d \in \Delta$ form an abelian group G_d under composition. They also constructed a surjective homomorphism from G_d to a subgroup of the ideal class group of R, identifying the kernel as those classes containing a form that represents a unit in D.

Also in 1968, Kaplansky [Kap68] extended the theory of composition to Bézout domain of characteristic not 2 via module multiplication. This recovers the same composition of united forms.

In 1972, Butts–Dulin [BD1972] studied the connection between Gauss composition and composition of united forms. They considered the compound and the Gaussian compound of binary quadratic forms defined by Gauss. Then they gave necessary and sufficient conditions for existence in each case over a Bézout domain and for existence of composition of united forms in an integral domain of characteristic not 2 where two primitive binary quadratic forms of the same discriminant has a Gaussian compound. Furthermore, they showed that the composition of united forms holds over a Bézout.

In 1980, Towber [Tow80] gave a generalization in which the domain of a quadratic form $Q\colon M \to R$ where M is a locally free module of rank 2 over a commutative ring R (with 1) such that $\bigwedge^2 M$ (equipped with an orientation given by a choice of generator) is a free

R-module of rank 1. He also introduced a composition law on the set of equivalence classes under the action of $\mathrm{SL}_2(R)$ of primitive, oriented binary quadratic forms $ax^2 + bxy + cy^2$ with a given discriminant and a given residue class of b in $R/2R$.

In 1982, Kneser [Kne82] considered quadratic forms $Q\colon M \to R$ over a commutative ring R where $\mathrm{Clf}^0(Q)$ is isomorphic to a given *type* which is an R-algebra C. This C-module structure ensures the existence and the uniqueness of a composition map. The isomorphism classes of primitive binary quadratic forms of type C with the composition law form an abelian group $G(C)$. Forgetting the quadratic form on M yields a homomorphism $G(C) \to \mathrm{Pic}(C)$ which is in general neither injective nor surjective. To obtain the full Picard group, he then considered M as invertible C-modules. [Kne82, Proposition 2] gives quadratic R-modules (also called *quadratic maps* by Kneser) $Q\colon M \to N$ where N is invertible R-module, $Q(cx) = \mathrm{Nm}_{C/R}(c)Q(X)$ for all $c \in C$ and $x \in m$, and Q is primitive. The isomorphism classes (oriented similarity classes in our notion) of such quadratic modules are isomorphic to $\mathrm{Pic}(C)$. This is exactly recovered by Equation (1.3.2).

Most of the works above only considered SL_2-equivalence of binary quadratic forms. In 2000, Mastropietro [Mas2000] considered equivalences given by matrices of determinants which are totally positive units over real quadratic number fields of class number one with totally complex quadratic extensions. He gave the contruction of a correspondence between ideal class groups and such equivalence classes.

In 2004, Bhargava [Bha04a] considered the space of $2 \times 2 \times 2$ cubical integer matrices modulo the natural action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ and described six composition laws, including Gauss composition, derived from this approach. He then gave interpretations of the connection between orbits of the six spaces and ideal classes of quadratic orders. (It would be interesting to pursue a sheafified version of this approach.) [Bha04b] and [Bha04c] in the same year and [Bha08] in 2008 developed analogous laws of composition on forms of degree $k > 2$ so that the resulting orbits parametrize orders in number fields of degree k.

In 2011, Wood [Woo11] gave a generalization of Gauss composition over any base scheme. Her dual point view considers linear binary quadratic forms over a scheme X as global sections of $\mathrm{Sym}^2(\mathscr{M}) \otimes \mathscr{L}$ where $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module of rank 2 and $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module (see discussion in section 5.3). The $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_1(Z)$-equivalences are exactly our notion of similarities. On the other hand, she considers pairs $(\mathscr{O}_Y, \mathscr{I})$ where $\mathscr{O}_Y$ is a quadratic $\mathscr{O}_X$-algebra and $\mathscr{I}$ is a traceable $\mathscr{O}_Y$-module. Equation (3.2.3) has shown that being traceable is equivalent to being pseudoregular. Then eq. (1.3.3) recovers [Woo11, Theorem 1.4], and thus restricting to primitive forms recovers [Woo11, Theorem 1.5] because primitive forms correspond to invertible modules by Equation (6.2.2). [Woo11, section 5] further equips an $\mathscr{L}$-type orientation corresponding to $(\mathscr{O}_Y, \mathscr{I})$, and Equation (1.5.1) recovers [Woo11, Theorem 5.1]. Unfortunately, Wood did not give group laws or isomorphisms with Picard groups in our style; instead, her results only provide a set-theoretical bijection with a disjoint union of quotients of Picard groups in the primitive case.

Wood mentions that Lenstra suggested in a talk that Kneser's approach could yield a theorem in the style of [Woo11, Theorem 5.1] in the case when the forms are primitive and nondegenerate. He also suggested a Clifford algebra construction which could provide a theorem along the lines of [Woo11, Theorem 1.4].

In 2016, O'Dorney [O'Do16] generalized Bhargava's theory of higher composition law over arbitrary Dedekind domains. The parametrization of quadratic, cubic, and quartic algebras

as well as ideal classes in quadratic algebras extended Bhargava's reinterpretation of Gauss composition.

In 2021, Zemková [Zem21] generalized Mastropietro's work to the case where the base field is any number field of narrow class number one and gave explicit correspondence between the ideal class group and the equivalence classes of binary quadratic forms. She also gave a short overview of previous works in the introduction section.

In 2024, following Wood's approach with extra refinements, Dallaporta [Dal24] recovered from Wood's bijection an explicit bijection between the Picard group of a given quadratic algebra and a set of classes of primitive quadratic forms over a scheme. Equation (1.3.2) recovers [Dal24, Theorem 1.2] where the corresponding quadratic algebra is specified by the discriminant and parity.

In 2025, Bitan [Bit25] considered Gauss composition for binary quadratic forms over the ring of regular functions over an affine curve over a finite field of odd characteristic. He used étale cohomology to describe the bijection between the equivalence classes of binary quadratic forms and the Picard group of the corresponding quadratic algebra.

Finally and very recently, Mondal–Venkata Balaji [MB2025] shared a preprint which also adopts the point of view that Gauss composition can be seen through the lens of the Clifford functor. They prove functorial bijections (but not an equivalence of categories) for similarities and show that for similitude groups one can recover the Picard group when 2 is a nonzerodivisor. To show that the Clifford map is bijective on classes, they use the method of universal norms due to Bischel–Knus [BK94].

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