256A: ALGEBRAIC GEOMETRY

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The following are notes from a course taught by Robin Hartshorne intending to cover the first two chapters of his text *Algebraic Geometry*. Only the supplementary comments and examples are included.

§I.1: Affine Varieties

Example. If $k = \mathbb{R}$, $A = \mathbb{R}[x, y]$, then the variety $Z(y - mx - b)$ defines a variety, a line.

Note that $x^2 + y^2 = 1$ gives a circle, but $Z(x^2 + y^2 + 1) = \emptyset$ and $Z(x^2 + y^2)$ consists of a single point—this is because $\overline{\mathbb{R}} = \mathbb{C} \neq \mathbb{R}$.

If $k = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{F}_p$, think of the variety as contained in the algebraic closure $\overline{k}$ and do algebraic geometry in this affine space, then look for points over k.

Example. $\mathbb{R}^1$ in its usual topology has $\dim \mathbb{R}^1 = 0$ since the only irreducible subset Y is a point: if $a \neq b \in Y$, choose $a < c < b$, so that $\mathbb{R} = (-\infty, c] \cup [c, \infty)$, and one can intersect this with Y to obtain a decomposition of Y. This works more generally for any Hausdorff space: for any two points in a subset, find the corresponding open sets $U \cap V = \emptyset$ containing these points and take their complements.

For us, dimension is given by chains of distinct primes.

Definition. There are four equivalent ways to define the dimension of a ring:

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(1) For any ring R, we have the Krull dimension, which is
$$\dim R = \sup\{r : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \subset R\}$$
for distinct prime ideals of R.

(2) Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. Then we define
$$\dim A = \inf\{n : x_1, \dots, x_n \in \mathfrak{m},\ A/\langle x_1, \dots, x_n\rangle \text{ Artin}\};$$
recall that a ring is *Artin* if it is of finite length, i.e. there exists an upper bound for the length of chains of ideals (e.g. $k[x]/\langle x^2\rangle$).

(3) For A local, we define
$$\operatorname{gr}_{\mathfrak{m}} A = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1} = k \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \dots$$
where $k = A/\mathfrak{m}$ is the residue field. We have $\dim_k \mathfrak{m}^\nu/\mathfrak{m}^{\nu-1} < \infty$, denoted $\phi_A(\nu)$. Then there exists a polynomial p_A with rational coefficients such that for all sufficiently large ν, $\phi_A(\nu) = P_A(\nu)$. We set $\dim A = \deg P_A + 1$.

(4) For R an integral domain containing a field k, we consider $k \subset K(R)$ the field of fractions of R. Then $\operatorname{trdeg}_k K(R) = \dim R$.

We have the following:

Theorem. *If R is a finitely generated k-algebra, then $\operatorname{trdeg}_k K(R)$ is equal to the Krull dimension. If R is any noetherian ring, $\dim R = \sup_{\mathfrak{p} \subset R} \dim R_{\mathfrak{p}}$. If A is a local noetherian ring with maximal ideal $\mathfrak{m}$, then the definitions above agree with the Krull dimension.*

We can compute the dimension of $\mathbb{A}^n$ in many ways.

Example. We note that $\dim \mathbb{A}^n_k = n$ for $k = \overline{k}$. Let $A = k[x_1, \dots, x_n]$ so by definition $\dim \mathbb{A}^n_k = \dim A$. This follows now immediately since $K(A) = k(x_1, \dots, x_n)$ has transcendence degree n over k. Alternatively, we have $\langle 0\rangle \subset \langle x_1\rangle \subset \cdots \subset \langle x_1, \dots, x_n\rangle$ so $\dim A \geq n$. But the localization $A_{\langle x_1, x_2, \dots, x_n\rangle}$ when divided by $\langle x_1, \dots, x_n\rangle$ gives k which is Artin, so $\dim A \leq n$. Finally, $\operatorname{gr}_{\mathfrak{m}} A = A$, and $\phi(\nu)$ counts the number of monomials of degree ν in $x_1, \dots, x_n$, which totals $\binom{n+\nu-1}{n-1}$, which is a polynomial in ν of degree $n-1$.

§I.2: Projective Varieties

Here is a concrete description of projective space:

Example. The projective line $\mathbb{P}^1_k$ is the set of points $(a_0 : a_1)$ modulo $k^\times$. If $a_0 \neq 0$, we can take as a representative $(a_0 : a_1) = (1 : a_1/a_0) = (1 : b)$ for $b \in k$; if $a_0 = 0$, $a_1 \neq 0$ by definition so $(a_0 : a_1) = (0 : a_1) = (0 : 1)$. Therefore as a set,
$$\mathbb{P}^1_k = \{(1 : b) : b \in k\} \cup \{(0 : 1)\} = \mathbb{A}^1_k \cup \{\infty\}.$$

Similarly, the projective plane is $\mathbb{P}^2_k = \mathbb{A}^2_k \cup \mathbb{P}^1_k$, including the line at infinity.

Looking at projective versions of affine varities can lead to some very important (and surprising) information:

Example. We have $\mathbb{A}^2_{\mathbb{R}} \simeq U_0 \subset \mathbb{P}^2_{\mathbb{R}}$ where $U_0 = \mathbb{P}^2_{\mathbb{R}} \setminus Z(x_0)$. Therefore the conic $x_1^2 + x_2^2 = x_0^2$ is of the form $x^2 + y^2 = 1$ and does not intersect the line at infinity (as is plain from the graph).

Alternatively, the curve $C : y = x^2$ lifts to $\overline{C} : x_0 x_2 = x_1^2$, so $x_0 = 0$ implies $x_1^2 = 0$, and we have the single intersection point $(0 : 0 : 1)$. Looking in U_2, we see that the parabola is tangent to the line at infinity.

Finally, the seemingly honest curve $y = x^3$ has the projective closure $x_0^2 x_2 = x_3^3$, which looks on the set $x_2 \neq 0$ like $u^2 = v^3$, so the curve has a cusp at infinity!

Here is an extended description of the twisted cubic curve.

Example. The affine version of the twisted cubic curve C is the subset $\{(a, a^2, a^3) : a \in k\} \subset \mathbb{A}_k^3$, i.e. the set of points parameterized by $x = t$, $y = t^2$, $z = t^3$.

Claim. C is a closed, irreducible subset of $\dim 1$.

To see this, we find the prime ideal $\mathfrak{p} \subset k[x, y, z] = A$ defining this ideal. We map $A \xrightarrow{\psi} k[t]$ by $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$; since the image is a domain, $\ker \psi = \mathfrak{p}$ is prime. We guess that $\mathfrak{p} = I(C)$. If $f = f(x, y, z) \in \mathfrak{p}$, then f vanishes on C: $f(a, a^2, a^3) = \psi(f)(a) = 0$, so $C \subset Z(\mathfrak{p})$. Conversely, if $P = (a, b, c) \in Z(\mathfrak{p})$, then for all $f \in \mathfrak{p}$, $f(a, b, c) = 0$. Since $y - x^2$, $z - x^3 \in \mathfrak{p}$, $b = a^2$, $c = a^3$, so $P = (a, a^2, a^3)$. So $C = Z(\mathfrak{p})$, so C is certainly a closed and irreducible subset. In particular, $\dim C = \dim A/\mathfrak{p} = \dim k[t] = 1$, which proves the claim.

How many equations define C? Take $f_1 = y - x^2$, $f_2 = z - x^3$. Then $Z(f_1, f_2) = C$, since $\langle f_1, f_2 \rangle \subset \mathfrak{p}$, so $Z(f_1, f_2) \supset Z(\mathfrak{p}) = C$, but we have actually shown by the above equality just on these generators. How many equations define the *prime* $\mathfrak{p}$? Simply, $\mathfrak{p} = \langle f_1, f_2 \rangle$ since $A/\langle f_1, f_2 \rangle = k[x]$ already.

Now projectivize C: We have $\mathbb{A}^3 \simeq U_0 \subset \mathbb{P}^3$, $U_0 = \mathbb{P}^3 \setminus Z(x_0)$. Then $C \subset \mathbb{A}^n \subset \mathbb{P}^n \supset \overline{C}$. For any set V, if V is irreducible, then $\overline{V}$ is irreducible. Therefore $\overline{C}$ is a closed irreducible subset of $\mathbb{P}^n$ of dimension 1.

We homogenize $\mathfrak{p}$ directly and have $g_1 = yw - x^2$, $g_2 = w^2 z - x^3 \in \overline{\mathfrak{p}} = I(\overline{C})$. Do these equations define $\overline{C}$? No, because if $x = w = 0$, $L \subset Z(g_1, g_2)$, but $\overline{C}$ is not a line and is irreducible. We also have $y^2 - xz = g_3 \in \overline{\mathfrak{p}}$. We would like $\overline{C} = Z(g_1, g_2, g_3)$.

We know that $\overline{C} \subset Z(g_1, g_2, g_3)$. Next, if $P = (a : b : c : d) \in \overline{C}$, if P is an affine point ($d \neq 0$), then $P \in C$ by earlier work. Otherwise, $d = 0$, so $a = 0$ and then $b = 0$, so $P = (0 : 0 : 1 : 0) \in \mathbb{P}^3$. For the moment, we will omit the reason why $P \in \overline{C}$.

Instead, we ask if g_1, g_2, g_3 generate $\overline{\mathfrak{p}}$. If $g(x, y, z, w) \in \overline{\mathfrak{p}}$ we can substitute for the x^2, x^3, and y^2 terms, so what is left is of the form

$$h_1(z, w) + x h_2(z, w) + y h_3(z, w) + xy h_4(z, w).$$

We now take $k[x, y, z, w] \xrightarrow{\overline{\psi}} k[t, u]$ by $x, y, z, w \mapsto tu^2, t^2 u, t^3, u^3$. We find $g_4 = xy - zw \in \overline{\mathfrak{p}}$ which allows us to remove the h_4 term, and under this substitution the h_i are cubes (in t and u), so it must be identically zero. Since $x g_1 - w g_4 = g_2$, we have $\overline{p} = \langle g_1, g_3, g_4 \rangle$, so indeed $P \in \overline{C}$.

Claim. $\overline{\mathfrak{p}}$ cannot be generated by < 3 elements.

$\overline{\mathfrak{p}}$ is a homogeneous ideal so $S \supset \overline{\mathfrak{p}} = \bigoplus_{d=0}^{\infty} \overline{\mathfrak{p}}_d$. We have $\overline{\mathfrak{p}}_0 = 0$ and $\overline{\mathfrak{p}}_1 = 0$. $\overline{\mathfrak{p}}_2$ is the k-vector space generated by the qudratic polynomials $g_1, g_2, g_4 \subset S_2 = k\{x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2\}$, a space of dimension 10. We must have the g_i linearly independent over k, because dividing out by z, w, we find x^2, y^2, xy are linearly independent.

§I.3: Morphisms

Here are examples of regular functions:

Example. If we take the affine line $\mathbb{A}^1_k$, an open set $V \ni 0$, then f is regular if $f = g/h$ with $h(0) \neq 0$; h has finitely many zeros, so we can shrink the open set, and we find f is regular at 0 iff $f \in k[x]_{\langle x \rangle}$.

Example. If we take the projective line $\mathbb{P}^1$, we find $\mathscr{O}(V) = k$. For $U_0 = \mathbb{P}^1 \setminus Z(x_0) = \mathbb{A}^1$ with $\mathscr{O}(U_0) = k[x]$. But $U_1 \subset \mathbb{P}^1 \setminus Z(x_1) = \mathbb{A}^1$ with $\mathscr{O}(U_1) = k[y]$, $y = 1/x$. A function that is a polynomial in x and $1/x$ is constant.

Here is an alternative proof of:

Theorem (Theorem 3.2(a))**.** *$\mathscr{O}(Y) \simeq A(Y)$ when Y is affine.*

Proof. Let $f = g_i/h_i$ on the open set U_i, on any open cover such that $V = \bigcup_i U_i$. $h_i \neq 0$ on U_i and the U_i cover Y, so $Z(h_1, h_2, \dots) \cap V = \emptyset$, so $\sqrt{\langle h_1, h_2 \dots \rangle} = A(V)$ so $1 = \sum_{i=1}^r a_i h_i$ (the sum is finite) for certain $a_i \in A(V)$, and thus $f = \sum_{i=1}^r a_i f h_i = \sum_{i=1}^r a_i g_i \in A(V)$ since $f h_i = g_i$ on U_i which is dense in V (f is continuous). □

Let $\mathscr{C}$ be the category of varieties, with objects varieties and the arrows morphisms.

Example. $\mathbb{A}^1 \xrightarrow{\phi} Y = Z(y) \subset \mathbb{A}^2$ by $x \mapsto (x, 0)$ has $f(x,y)/g(x,y) = f(x,0)/g(x,0)$ regular on $\mathbb{A}^1$, therefore $A^1 \simeq Y$ since $\psi : Y \to \mathbb{A}^1$ by $(x, 0) \mapsto x$ is an inverse.

Example. The variety $Y = Z(y - x^2) \subset \mathbb{A}^2$ has $Y \simeq \mathbb{A}^1$ by $(x, x^2) \mapsto x$. To see this, we prove:

Lemma. *If V, W are affine varieties, and $\phi : V \to W$ a morphism, we have $\phi^* : A(W) = \mathscr{O}(W) \to \mathscr{O}(V) = A(V)$. ϕ is an isomorphism iff ϕ^* is an isomorphism.*

Proof. ϕ an isomorphism implies ϕ^* an isomorphism is true for any (not necessarily affine) V, W. Use correspondence: $P \in V \leftrightarrow \mathfrak{m}_P \subset A(V)$. We define $\psi : W \to V$ using the equivalence $Q \in W \leftrightarrow \mathfrak{m}_Q \subset A(W)$, since ϕ^* is an isomorphism, and thus ψ is bijective. The map is a homeomorphism because ψ^* takes ideals to ideals—just carry over quotients of functions. □

Returning to the example, we find $\phi^* : k[x,y]/\langle y - x^2 \rangle \to k[x]$ is an isomorphism, so ϕ is an isomorphism.

Here is some more category language. If $\mathscr{C}$ is the category of varieties, $\mathscr{O}$ is a contravariant functor from $\mathscr{C}$ to k-algebras (domains), since a map $V \to W$ induces a map $\mathscr{O}(W) \to \mathscr{O}(V)$. We have in fact that the subcategory of affine varieties mapping to the subcategory of finitely generated k-algebras is an equivalence of categories. For if A is such an algebra, $k[x_1, \dots, x_n] \to A$ by $x_i \mapsto a_i$ for generators a_i is surjective (if V is a variety defined by $\mathfrak{p}$ in $\mathbb{A}^n$, a different choice of a_i gives an isomorphic V).

Now for something really wild: If we take the subcategory of $\mathscr{C}$ of those varieties such that $\mathscr{O}(V)$ is a finitely generated k-algebra, we can look at the adjoint functor F. If V is a variety, $W_A = F(A)$ finitely generated with a map ϕ to $\mathscr{O}(V)$, if $P \in V$, we consider $\mathfrak{m}_P$, which is not necessarily a one-to-one map, but we can still have $\mathfrak{m}_Q \subset \phi^{-1}(\mathfrak{m}_P) \subset A$ maximal (look at the quotient fields), so we have $Q \in W_A$. In other words, $\mathrm{Hom}_{\mathscr{C}}(V, F(A)) = \mathrm{Hom}(A, \mathscr{O}(V))$, so the functors are adjoint. (Indeed, one can define affine varieties in this way.)

Example. It is possible to have $\phi : V \to W$ that is a bijective homeomorphism but is *not* an isomorphism.

The map $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (t^2, t^3)$ has $\phi^* : k[x,y]/\langle y^2 - x^3\rangle \to k[t]$ by $x, y \mapsto t^2, t^3$, so $\operatorname{img}\phi = V(y^2 - x^3)$. This map is bijective because $t = y/x$ is an inverse (except at $(0,0)$ which we send to zero). It is a homeomorphism because the Zariski topology is weak and all curves are homeomorphic. But $t \in \operatorname{img}(k[t] \xleftarrow{\phi} k[x,y]/\langle y^2 - x^3\rangle) = k[t^2, t^3]$; we pullback t to the function y/x which fails to be defined at 0.

Example. If $O \in \mathbb{P}^n$ is a fixed point, we have the morphism *projection from O* as a map $\mathbb{P}^n \setminus \{O\} \to \mathbb{P}^{n-1}$ as follows: take $O = (0 : \cdots : 0 : 1)$ for simplicity, and let $P = (a_0 : \cdots : a_n)$, $a_0, \dots, a_{n-1}$ not all zero. Then the line between O and P is $\lambda O + \mu P = (\mu a_0 : \cdots : \mu a_{n-1} : \mu a_n + \lambda)$, which when intersected with $\mathbb{P}^n \cap Z(x_n)$ gives a point $(a_0 : \cdots : a_{n-1})$. For example, taking $x, y, z, w = tu^2, t^2u, t^3, u^3$, if one projects from the point $(0 : 0 : 1 : 0)$ one obtains the cuspidal cubic.

(The only morphism $\mathbb{P}^n \to \mathbb{P}^{n-1}$ is constant, as we shall see later.)

Even when $k = \overline{k}$, $\mathbb{P}^n_k \supset V \supset U$ open, $\mathscr{O}(U)$ is not necessarily noetherian.

Here is a rather interesting consequence of the fact that there are no global regular functions on a projective variety:

Example. Every two quadrics in $\mathbb{P}^2$ have a nontrivial intersection. Let $U = \mathbb{P}^2 \setminus C_1$. If $C_1 \cap C_2 = \emptyset$, $C_2 \subset U$. By (3.4), $\mathscr{O}(C_2) = k$. Let $C_1 = Z(f)$ where f is of degree d. Then the functions $x_0^d/f, x_1^d/f, x_2^d/f$ are regular on C_2. If $P, Q \in U$ are distinct points, then there exists a linear combination g/f (with g of degree d) of these that satisfies $(g/f)(P) \neq (g/f)(Q)$, so g/f is a nonconstant regular function on C_2, which is a contradiction.

Here is another bit of category theory: For any category $\mathscr{C}$, we can define the *product* Z of X and Y to be an object equipped with morphisms $p : Z \to X$ and $q : Z \to Y$ with the universal property that for all W with $f : W \to X$ and $g : W \to Y$, there exists a unique $\theta : W \to Z$ such that $f = p \circ \theta$ and $g = q \circ \theta$. If this object exists, it is unique up to unique isomorphism.

In the category of affine varieties, products exist: One takes $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, and $Z = \{P : p(P) \in V,\ q(P) \in W\} \subset \mathbb{A}^{n+m}$. If $A = k[x_1, \dots, x_n] \supset I(V)$ and $B = k[y_1, \dots, y_m] \supset I(W)$, then $I(Z) = I(V)C + I(W)C$ where $C = k[x_1, \dots, x_n, y_1, \dots, y_m]$ is the compositum. (Or, we can just take $A(V) \otimes_k A(W)$ and verify that each satisfies the same universal property.) We write $Z = V \times W$, but the Zariski topology on Z is *not* the induced product topology.

§I.4: Rational Maps

The idea of birationality is to generalize the idea that the maps $\mathbb{A}^1 \xrightarrow{f} \mathbb{P}^1$ together with the map $U_0 \xrightarrow{g}$ $(U_0 = \mathbb{P}^1 \setminus Z(x_0))$ are "inverse" when restricted to these dense open sets. We say that $\mathbb{A}^1$ is birationally equivalent to $\mathbb{P}^1$.

Remark. For curves, birational equivalence does not see singularities (because they are of lower dimension, hence contained in a finite set), but does see the genus g of the curve. For $g = 0$, we just have $\mathbb{P}^1$; for $g = 1$ (elliptic curves), the j-invariant classifies curves (over $k = \overline{k}$) up to birational equivalence.

Here is a concise proof of the affine cover proposition:

Proposition (Lemma 4.2 and Proposition 4.3)**.** *For all $P \in X$ and all $V \ni P$ open in X, there exists an affine open U such that $P \in U \subset V$.*

Proof. If X is open in $\mathbb{P}^n$, we have an affine cover $U_0, \dots, U_n$, so for $P \in U_i$ we consider $U_i \cap X$, so it suffices to treat quasi-affine or affine varieties. Since $X \subset \overline{X} \subset \mathbb{A}^n$, we can (again by intersecting) treat the affine case.

Consider $Z = X \setminus V$. Z is closed in X so is closed in $\mathbb{A}^n$. $P \notin Z$, so there exists $f \in I(Z)$ such that $f(P) \neq 0$, so $Z \subset Z(f) \not\ni P$. Let $U = X \setminus Z(f) = X \cap (\mathbb{A}^n \setminus Z(f))$. Thus it is enough to show that $\mathbb{A}^n \setminus Z(f)$ is affine.

We look in $\mathbb{A}^{n+1}$, and define $g = 1 - f(x_1, \dots, x_n)x_{n+1}$. We have $\mathbb{A}^n \setminus Z(f) \simeq Z(g) \subset \mathbb{A}^{n+1}$: we map

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 1/f(a_1, \dots, a_n))$$

when $f(a_1, \dots, a_n) \neq 0$, with the inverse map $(b_1, \dots, b_{n+1}) \mapsto (b_1, \dots, b_n)$ (which is defined as $1 - f(b_1, \dots, b_n)b_{n+1} = 0$ so $f(b_1, \dots, b_n) \neq 0$). □

Here is another proof of:

Theorem (Corollary 4.5)**.** *For any two varieties X and Y, the following conditions are equivalent:*

(i) *X and Y are birationally equivalent;*
(ii) *There exists open sets $U \subset X$ and $V \subset Y$ such that $U \simeq V$ as varieties;*
(iii) *$K(X) \simeq K(Y)$ as k-algebras.*

Proof. (ii) $\Rightarrow$ (i): The isomorphism $\phi : U \simeq V$ gives a birational equivalence.

(i) $\Rightarrow$ (iii): We have a contravariant functor from the category of varieties with morphisms birational maps to the category of finitely generated fields over k. By the properties of functors, an isomorphism in the first category becomes an isomorphism in the second.

(iii) $\Rightarrow$ (ii): Given X, Y with $K(X) \simeq K(Y)$, we may assume that X and Y are affine—$K(U) = K(X)$ for an open U, since if $f, g : X \to Y$ are morphisms with $f|_U = g|_U$ then $f = g$ as U is dense in X. Let $Y \subset \mathbb{A}^n$, $y_1, \dots, y_n \in K(Y)$; then $x_i = \phi(y_i)$ are rational on X, and thus there exists a $U \subset X$ open such that $x_1, \dots, x_n$ are regular (take $U = \bigcap_{i=1}^n U_i$ where x_i are defined on U_i by removing the closed set of points where x_i is not defined). We have a map $\psi : U \to Y$ by $P \mapsto (x_1(P), \dots, x_n(P))$. By symmetry, we have a map $\phi : V \to X$ where $V \subset Y$ is open. These map induce the isomorphism $K(X) \simeq K(Y)$ by construction, so they are inverse where they are defined. If we consider $\phi^{-1}(V) \subset U$, we have that $\psi \circ \phi : \phi^{-1}(V) \to X$ is the identity as a rational map, similarly $\phi \circ \psi : \psi^{-1}(U) \to Y$. But since ϕ and ψ are inverse, we actually have that $\phi : \phi^{-1}(V) \to \psi^{-1}(U)$ gives an isomorphism on these open sets. □

Here are examples of birational maps.

Example. We have $\mathbb{A}^n \sim \mathbb{P}^n$ ($X \sim Y$ is notation for X birational to Y).

The map $\mathbb{A}^1 \to Y$ by $t \mapsto (t^2, t^3)$ is a birational map (where $Y = Z(x^3 - y^2)$ is the cuspidal cubic). This is because $\mathbb{A}^1 \setminus \{0\} \simeq Y \setminus \{(0,0)\}$, the inverse map being given by $(x, y) \mapsto y/x$.

Definition. A variety X is *rational* if $X \sim \mathbb{P}^n$ for some n.

Example. The nodal cubic curve $Y : y^2 = x^2(x+1) \subset \mathbb{A}^2$ is rational. We consider Y in $\mathbb{P}^2$ with coordinates x_0, x_1, x_2 with $x = x_1/x_0$, $y = x_2/x_0$. The projection map $\pi : \mathbb{P}^2 \setminus \{(0:0:1)\} \to L_\infty \simeq \mathbb{P}^1$ gives a map $\pi : Y \setminus \{(0:0:1)\} \to \operatorname{img} \pi \subset L_\infty$, which we claim is in fact an isomorphism.

Take the line $y = mx$ and a point $P \in Y \setminus \{(0:0:1)\}$. The intersection of the line and Y has $y^2 = x^2(x+1) = m^2x^2$ so $0 = x^2(x - (m^2 - 1))$. So since

$x \neq 0$, we have $x(P) = m^2 - 1$, and thus $P = (m^2 - 1 : m(m^2 - 1) : 1)$. We have $L_\infty = Z(x_0)$, so $P \mapsto m$ gives a morphism $\mathbb{A}^1 \setminus \{(0:0:1)\} \simeq \mathbb{A}^1 \setminus \{\pm 1\}$ with inverse $m \mapsto (m^2 - 1 : m(m^2-1) : 1)$. (The absence of ± 1 correspond to the slopes of the tangent lines of the curve at the origin.)

In fact, any irreducible cubic curve in $\mathbb{P}^2$ is either cuspidal or nodal (hence rational) or elliptic (which we will see later is *not* rational).

Example. Take the (nonsingular) quadric surface $Q \subset \mathbb{P}^3$ given by the equation $xy = zw$. We will show that $Q \sim \mathbb{P}^2$.

Let $O = (0:0:0:1)$, $H = Z(w)$. Our projection map π takes $(a:b:c:d) \mapsto (a:b:c) \in \mathbb{P}^2$. The map θ by $(a:b:c) \mapsto (a:b:c:ab/c)$ is an inverse if we restrict to the open set U where $c \neq 0$. In other words, we have $\pi : Q \setminus \{O\} \simeq \mathbb{P}^2 \setminus Z(z)$ with inverse θ.

We can understand this map better. Inside Q we have the line $L : Z(x,z) \subset Q$. If $P \in L$, then $P = (0:b:0:d) \mapsto (0:b:0) = (0:1:0)$, so the line $L \setminus \{O\}$ is squashed by π to the point $\{(0:1:0)\}$; the line $M : Z(y,z) \subset Q$ is also collapsed. So the variety $Z(z)$ has a whole line collapsed down into it, but the points $(a:b:0)$, $ab \neq 0$ are not in the image, i.e. $Z(z) \setminus \{(0:1:0),(1:0:0)\}$ has no preimage; this is the "tangent line" as $P \rightsquigarrow O$.

The process of "blowing up" is also referred to as monodial transformation, σ-process, dilitation, locally quadratic transformation, and *éclatement* or *aufblasen*.

Here is a concrete illustration of blowing up: the plane $\mathbb{A}^2$ at the origin $P = (0,0)$. We take the space $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates $(x, y; t : u)$. The blowup is the subvariety Y defined by $xu - yt$, which is homogeneous in t, u, and we have the projection map $\pi : Y \to \mathbb{A}^2$.

Inside Y we have the line $x = y = 0$; this is $E \simeq \mathbb{P}^1$ with $\pi^{-1}(P) = E$. If $Q \in \mathbb{A}^2$ with $Q \neq P$, then $Q = (a,b)$ with a, b not both zero. Then

$$\pi^{-1}(Q) = \{(a, b; t : u) : au = bt\} \subset \mathbb{A}^2 \times \mathbb{P}^1;$$

if $b \neq 0$ then $t = au/b$ so $\pi^{-1}(Q) = \{(a, b : au/b : u)\}$; since $u = 0$ implies $t = 0$, which is not a point of the projective line, we have the point $\pi^{-1}(Q) = \{(a, b : a/b : 1)\} = \{(a, b : a : b)\}$. A similar argument shows the same if $a \neq 0$. Therefore we have that

$$\begin{aligned} Y \setminus E &\to \mathbb{A}^2 \setminus \{P\} \\ (a, b; a : b) &\mapsto (a, b) \end{aligned}$$

is an isomorphism.

Y is irreducible: One could either develop a theory of varieties in $\mathbb{A}^n \times \mathbb{P}^r$ with appropriate homogeneity conditions, but it also suffices to argue: since $Y = (Y \setminus E) \cup E$, and $Y \setminus E$ is irreducible, we need only check that every point of E is in the closure of $Y \setminus E$. Let $L : y = mx$ be a line in $\mathbb{A}^2$: $\pi^{-1}(L)$ is the set of simultaneous solutions to $y = mx$ and $xu = yt$, so $xu - mxt = 0 = x(u - mt)$, thus $\pi^{-1}(L) = E \cup \widetilde{L}$, with $\widetilde{L} \cap E = \{Q\} = \{(0, 0 : 1 : m)\}$. Together with the vertical line, this shows that the closure of Y contains L and its closure, which contains all of E.

We can generalize the blowup as follows:

Definition. If $V \subset \mathbb{A}^n$ is any variety, we define the *blowup of V at P*, also called the *proper transform of V* to be $\overline{\pi^{-1}(V \setminus P)}$.

At first glance it seems as though this depends on the embedding, but we will see later that it is in fact independent.

§I.5: Nonsingular Varieties

As a motivation for the definition of nonsingularity, we consider affine plane curves C defined by $f(x, y) = 0$. Write $f = f_0 + f_1 + f_2 + \cdots + f_d$ in its homogeneous parts. Assume that $P = (0, 0)$ is on the curve, so that $f_0 = 0$. Then very near to P, any higher degree term is negligible in comparison to a lower term, therefore a curve which is smooth near P must have only a single tangent direction, hence C is nonsingular at P iff $f_1 \neq 0$. Since $f_1 = f_x(0, 0)x + f_y(0, 0)y$, as one can calculate, this is to say that C is smooth at P iff $f_x(P), f_y(P)$ are not both zero.

As an example of regular local rings, we have:

Example. The rings $A = k[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}$ and $A = k[[x_1, \ldots, x_n]]$ each with maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ are regular, since $\mathfrak{m}/\mathfrak{m}^2 = kx_1 \oplus \cdots \oplus kx_n$, which has dimension $n = \dim A$, that is because $x_1, \ldots, x_n \in \mathfrak{m}$ is a regular sequence.

In the proof of Theorem 5.1, it is cleaner if one assumes after a change of variables that the point is $P = (0, \ldots, 0)$ and hence $\mathfrak{m}_P = \langle x_1, \ldots, x_n\rangle$.

Here are three more calculations of the singular locus:

Example. Consider the Fermat curve $f = x^3 + y^3 + z^3 = 0$ in $\mathbb{P}^2$. By symmetry, it suffices to consider the points where $z = 1$, and we have the affine model $f = x^3 + y^3 + 1$. Hence $f_x = 3x^2$, $f_y = 3y^2$, so there are no singular points if $\operatorname{char} k \neq 3$. In the case that the characteristic is 3, then $x^3 + y^3 + z^3 = (x + y + z)^3$, so $Z(f) = Z(x + y + z)$, and this is also nonsingular.

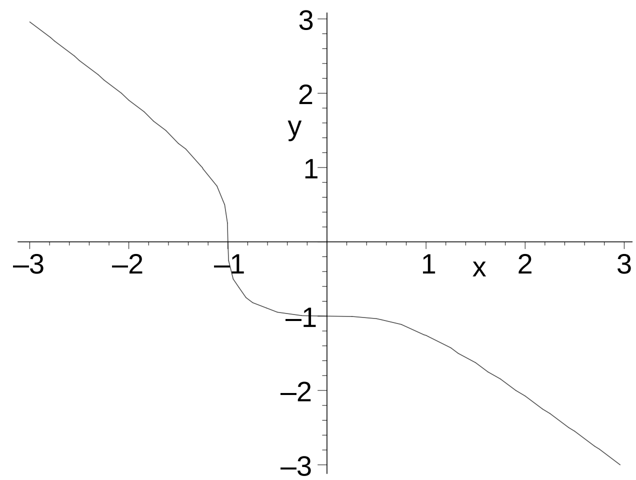

FIGURE 1. $x^3 + y^3 + 1 = 0$

Example. Consider the surface $f = x^2 - yz$ in $\mathbb{A}^3$. We have $f_x = 2x$, $f_y = -z$ and $f_z = -y$. Hence $y = z = 0$, and thus $x = 0$, and the only singular point is $(0, 0, 0)$. This surface is normal, so is an example that normal does not imply nonsingular (though nonsingular implies normal).

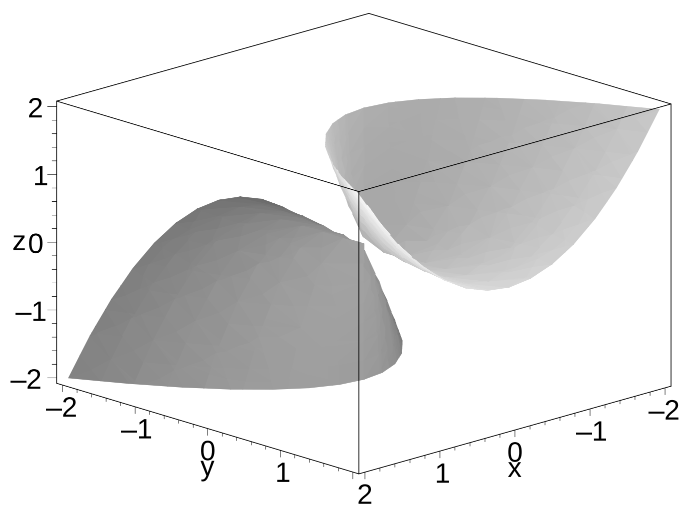

FIGURE 2. $x^2 - yz = 0$

Example. Consider $V : x^2z - y^2w$ in $\mathbb{P}^3$. When $w = 1$, $x^2z - y^2 = f$, $f_x = 2xz$, $f_y = -2y$, $f_z = x^2$, so we conclude that there is a line of singular points $L : x = y = 0$. If we let $z = a^2$, then $y = \pm ax$, and the surface "passes through itself".

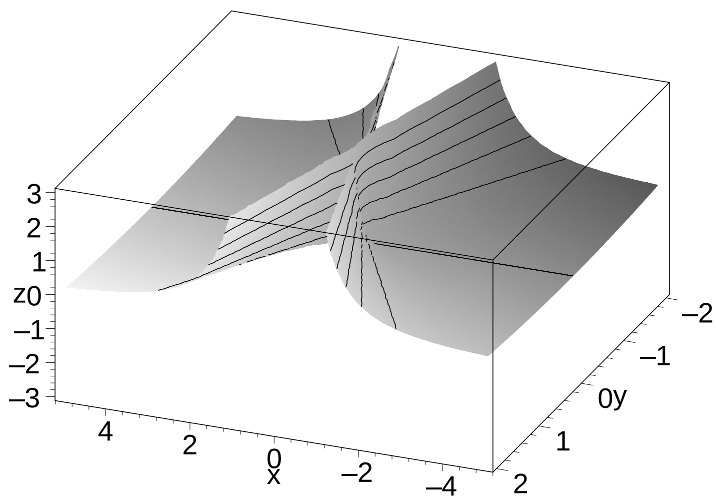

FIGURE 3. $x^2z - y^2 = 0$

We can also compute the singular locus of the Legendre family of elliptic curves:
Example. We compute for $\lambda \in k$ the singular locus of the family of elliptic curves

$$C : y^2 = x(x-1)(x-\lambda) = x^3 - (\lambda+1)x^2 + \lambda x.$$

We have $f_y = 2y = 0$ implies $y = 0$ whenever $\operatorname{char} k \neq 2$, and this implies that $x = 0, 1, \lambda$ but $f'(x) \neq 0$ at these points unless $\lambda = 0, 1$. Therefore we have three cases: either C is nonsingular, it has a node (when $\lambda = 1$) or it has a double point at the origin (when $\lambda = 0$).

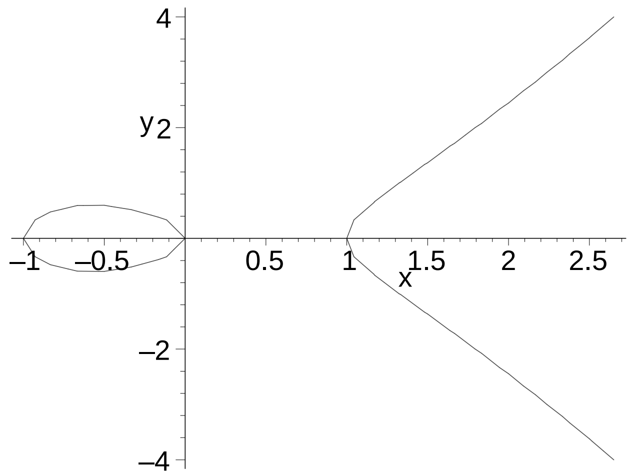

FIGURE 4. Nonsingular elliptic curve ($\lambda \neq 0, 1$)

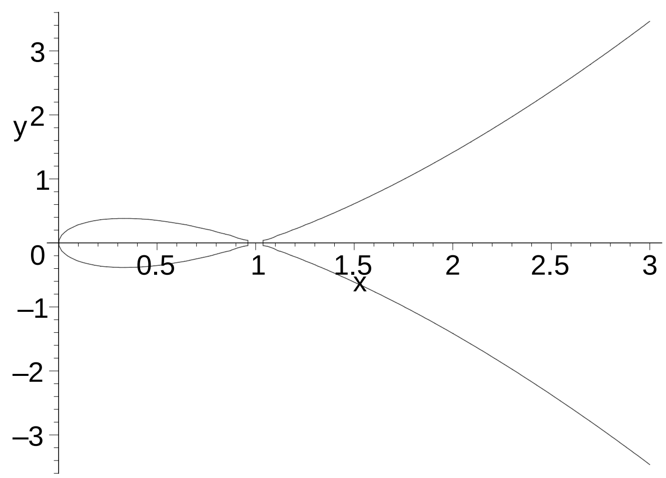

FIGURE 5. Nodal cubic ($\lambda = 1$)

One would also like to know what happens at infinity. We can projectivize to obtain $y^2z = x(x-z)(x-\lambda z)$, so $z = 0$ implies $x^3 = 0$, so $y = 1$, and in this neighborhood the equation takes the form $z = x(x-z)(x-\lambda z)$ which is linear in z, so this point is nonsingular.

Here is another example of an elliptic curve which shows that one really must attend to characteristic:

Example. Consider $y^2 + y = x^3 - x$ in $\mathbb{A}^2$. We have $f_y = 2y + 1$, $f_x = 3x^2 - 1$, so if $\operatorname{char} k \neq 2, 3$ (in these cases the curve is nonsingular because $f_y = 1 \neq 0$, $f_x = -1 \neq 0$, respectively), we have $y = -1/2$ and $x^2 = 1/3$. Substituting we have $1/4 - 1/2 = x(x^2) - x = 1/3x - x = -2/3x = -1/4$, so $x = 3/8$, and thus $(3/8)^2 = 9/64 = 1/3$ iff $27 = 64$, i.e. $\operatorname{char} k = 37$. In this case, we have $3(5^2) \equiv 1$

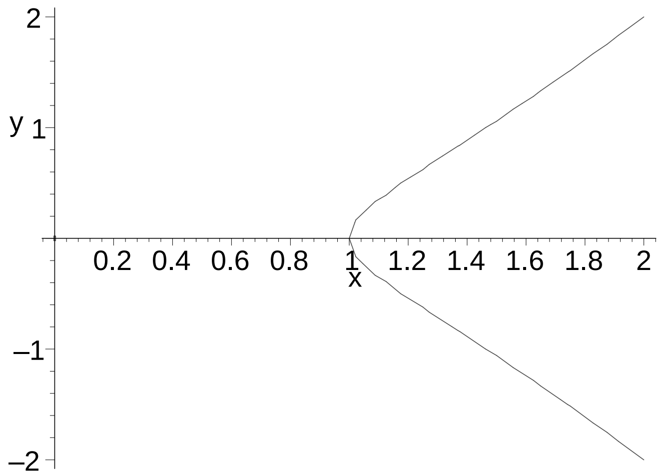

FIGURE 6. Cubic with a double point ($\lambda = 0$)

(mod 37) and $-2(18) \equiv 1 \pmod{37}$, and the point $(-5, 18)$ is not on the curve, so we can bring the singularity $(5, 18)$ to the origin by

$$(y' + 18)^2 + (y' + 18) = (x' + 5)^3 - (x' + 5), \qquad y'^2 = x'^3 + 15x'^2$$

which is a node.

Remark. In the course of proving Theorem 5.3, one also has proven: If $k = \overline{k}$ and A is a finitely generated domain over k, and $X = \operatorname{Specm} A$ (the set of maximal ideals of A) with the Zariski topology ($Z \subset X$ is closed if there exists an ideal $I \subset A$ such that $Z = \{\mathfrak{m} \in \operatorname{Specm} A : \mathfrak{m} \supset I\}$), then

$$X_{\text{sing}} = \{\mathfrak{m} \in \operatorname{Specm} A : A_{\mathfrak{m}} \text{ is not regular}\}$$

is closed, and $X_{\text{reg}} = X \setminus X_{\text{sing}}$ is a nonempty, open set.

This problem is called *the problem of the closedness of the singular set*, and it is not true for a general noetherian ring. (There is a counterexample due to Nagata.)

Here is an important counterexample:

Example. For part (5.4A(b)), it is important that M be finitely generated. Consider $A = k[x]_{\langle x \rangle}$ with maximal ideal $\mathfrak{m} = \langle x \rangle$ and quotient field $K = k(x)$. Then $\widehat{A} = k[[x]]$, but

$$K \otimes_A \widehat{A} = k[[x]][1/x] = k((x)),$$

whereas $\widehat{K} = \varinjlim K/\mathfrak{m}^n K = 0$.

Here is a more detailed exposition of:

Claim. If $A = k[[x, y]]$, $\phi : A \to A$ by $\phi(x) = g = x + \dots$, $\phi(y) = h = y + \dots$, then ϕ is an isomorphism.

Proof. If $0 \neq f \in \ker \phi$, then $f = f_d + \dots$ with $f_d \neq 0$, so by looking at degrees, $\phi(f) = f_d + \dots = 0$, so $f_d = 0$, a contradiction. Thus ϕ is injective.

To show ϕ is surjective, we note that if $f \in A$, with $f = f_d + \dots$, then $\phi(f_d) - f_d$ begins $f'_{d'} + \dots$ with $d' \geq d$, so we may proceed inductively to define a preimage. $\square$

Remark. Here are some facts about analytic isomorphisms ($\widehat{\mathscr{O}}_{P,V} \simeq \widehat{\mathscr{O}}_{Q,W}$):

(i) Any two nodes on curves are analytically isomorphic. The proof of Example 5.6.3 essentially shows this.
(ii) If $P \in V$, $Q \in W$ are analytically isomorphic, then $\dim V = \dim W$. This follows from the Hilbert-Samuel polynomial and the fact that $\dim \mathscr{O}_{P,V} = \dim V$ and $\operatorname{gr}_{\mathfrak{m}} A \simeq \operatorname{gr}_{\widehat{\mathfrak{m}}} \widehat{A}$.
(iii) Any two points of a nonsingular variety are analytically isomorphic. By the Cohen structure theorem, if $P \in V$ is nonsingular, $k \subset \mathscr{O}_{P,V}$ is a regular local ring, so $\widehat{\mathscr{O}}_{P,V} \simeq k[[x_1, \dots, x_n]]$ where $n = \dim V$.

Here is an alternative definition of multiplicity (compare (Ex. 5.3)) which is more general:

Definition. Let $P \in V$ of dimension n, look at $\mathscr{O}_{P,V}$. The function $\phi(\nu) = \dim_k \mathfrak{m}^\nu/\mathfrak{m}^{\nu+1}$ has the property that there exists a polynomial $P(z) \in \mathbb{Q}[z]$ such that $p(\nu) = \phi(\nu)$ for sufficiently large ν (the Hilbert-Samuel polynomial). We have $\deg p = \dim \mathscr{O}_P - 1$, and if we write $p(z) = a_0 z^{n-1} + \cdots + a_{n-1}$, then the multiplicity of P on V is $(n-1)!a_0$.

In the case of plane curves, the Hilbert polynomial is equal to d for sufficiently large d where $f(x, y) = f_d + \dots$, so this definition agrees with the one given in the exercises.

§I.6: Nonsingular Curves

The problem of classification of varieties up to isomorphism, even if we restrict to nonsingular varieties, is already very complication. Take the example of rational curves, which includes $\mathbb{A}^1$, $\mathbb{P}^1$, a conic in $\mathbb{A}^2$, $\mathbb{A}^1 \setminus \{P\}$ (all of which are isomorphic by a translation) $\mathbb{A}^1 \setminus \{P, Q\}$ (isomorphic by a dilation and translation), $\mathbb{A}^1$ minus three points (for which there is a one-parameter family; if we view this as $\mathbb{P}^1$ minus four points, then the four points have a cross-ratio which is invariant under linear fractional transformation). This is why we look first at varieties up to birational equivalence.

We would like to show that any curve C is birational to a nonsingular projective curve, but it is hard to obtain each of these simultaneously. Here are three approaches:

(1) One can blow up a projective embedding of C. Using plane Cremona transformations, one can obtain ordinary singular points, but this cannot be discussed here.
(2) If $C \subset \mathbb{A}^n$ is singular, $A(C) = A/I(C)$ the corresponding one-dimensional integral domain, we consider its normalization $\widetilde{A(C)} = B \subset K$. B is again a finitely generated integral domain over K and thus corresponds to a curve $\overline{C}$, which is then nonsingular (if A is a local noetherian ring of dimension 1, then A is normal iff it is regular). So given a singular projective curve we can take a finite open cover, normalize on each open piece, and glue (thanks to the universal property of the blowup). But this is already very tricky.
(3) We look at *abstract* nonsingular curves; this is the approach taken in the text.

In comparison to the situation of curves, we have the following fact:

Fact. *Any abstract curve is quasi-projective. Any abstract* nonsingular *surface is quasi-projective. But there exists an abstract nonsingular variety of dimension* *three which is not quasi-projective (due to Hironaka)—one blows up two curves in different orders on a surface.*

The important statement to take away from the equivalence of categories (6.12) is that any two *projective nonsingular* curves are birationally equivalent if and only if they are already isomorphic.

Here is an extended discussion of one of the homework problems (Ex. 6.2).

Example. We will show that $C : y^2 = x^3 - x$ is not rational. This is equivalent to showing that the quotient field L of $k[x,y]/\langle y^2 - x^3 + x\rangle$ is not a pure transcendental extension of k, though this seems to be an intractable algebra problem. Instead, we look at the following invariant: for $\mathbb{P}^1$, if U is any open affine subset corresponding to A over k, then U is a UFD. For the curve C, there exists an open affine A which is not a UFD; therefore if they were birational, they would have isomorphic open subsets, which is a contradiction.

The quotient field $L = k(x)[y]/\langle y^2 - x^3 + x\rangle$ is an extension of $K = k(x)$ of degree 2. The Galois action of L/K is $\sigma : y \mapsto -y$, cyclic of order 2. Every element of L can be written $\alpha = a(x) + b(x)y$, hence $\sigma(\alpha) = a(x) - b(x)y$, and we have the norm

$$N(\alpha) = \alpha\sigma(\alpha) = a(x)^2 - (x^3 - x)b(x)^2.$$

The norm is multiplicative, as $N(\alpha\beta) = (\alpha\beta)\sigma(\alpha\beta) = N(\alpha)N(\beta)$. We note that $B = k[x,y]/\langle y^2 - x^3 + x\rangle$ is an integral extension of $A = k[x]$. We want to show that B is integrally closed; for $P \in C$, this is equivalent to showing that P is nonsingular. We find $\partial f/\partial y = 2y = 0$ so $y = 0$ (assume $\operatorname{char} k \neq 2$), hence $x = 0, -1, 1$ and $\partial f/\partial x = -3x^2 + 1 \neq 0$ at these points.

The next claim is that $\alpha \in B$ is a unit iff $N(\alpha)$ is a unit in A; this is because if $\alpha\beta = 1$ then $N(\alpha)N(\beta) = N(1) = 1$, and if $N(\alpha) = \alpha\alpha^\sigma \mid 1$ then $\alpha \mid 1$.

The equation $y^2 = x(x-1)(x+1)$ looks like an equation of non-unique factorization. Indeed, $k[x]^\times = k$, and $a(x)^2 - (x^3 - x)b(x)^2 \in k$ happens only if $b(x) = 0$ (since the latter has odd degree), hence $B^\times = k$ as well. We know that y is irreducible because if y can be written as the product of two factors then so can its norm, $-x(x-1)(x+1)$, so there exists a norm from B which is of degree 1—which we see from the equation is impossible.

§I.7: Intersections in Projective Space

We will give an alternative proof of:

Theorem. *For $V \subset \mathbb{P}^n$, there exists a (unique) polynomial $P_V(z) \in \mathbb{Q}[z]$ such that $P_V(l) = \phi_V(l)$ for all $l \gg 0$.*

Recall that the Hilbert function is $\phi_V(l) = \dim_k S(V) \in \mathbb{Z}_{\geq 0}$. The reason that we only have $l \gg 0$ is because a polynomial cannot vanish identically for $l < 0$ and in general there is small range where $P_v(l) \neq \phi(l)$ due to subtleties.

We generalize this with $\phi_M(l) = \dim_k M_l \in \mathbb{Z}_{\geq 0}$ as follows:

Proposition (Theorem 7.5)**.** *For any finitely generated graded S-module M, there exists a polynomial $P_M \in \mathbb{Q}[z]$ such that $P_M(l) = \phi_M(l)$ for $l \gg 0$.*

Proof. If $n = 0$, $S = k[x_0]$ so if M is a finitely-generated graded $k[x_0]$-module, then by the structure theorem for modules over a PID, M is a finite direct sum $M = \bigoplus_i M_i$ where each M_i is equal to S or $S/\langle x_0 - a\rangle^r$. But due to the grading, the latter cannot occur unless $a = 0$, so $\phi_M = \sum_i \phi_{M_i}$ with $M = S$ or $M = S/\langle x_0^r\rangle$, and

$\phi_{M_i}(l) = 1$ and $\phi_{M_i}(l) = 0$, respectively, which matches, so the Hilbert polynomial is constant.

In general, $S = k[x_0, \ldots, x_n]$, M a finitely-generated graded S-mdoule. We have the exact sequence

$$0 \to Q \to M \xrightarrow{x_n} M \to N \to 0$$

for modules Q and N which are both annihilated by x_n and hence are graded $k[x_0, \ldots, x_{n-1}]$-modules, so by induction ϕ_Q and ϕ_N are represented by polynomials P_Q and P_N for $l \gg 0$.

Looking at degrees, we have

$$0 \to Q_{l-1} \to M_{l-1} \xrightarrow{x_n} M_l \to N_l \to 0$$

so as finite dimensional vector spaces, $\phi_M(l) - \phi_M(l-1) = \phi_N(l) - \phi_Q(l-1) = P_M(l) - P_Q(l-1)$ for $l \gg 0$. Therefore by the difference lemma (7.3), ϕ_M is also a polynomial, and we are done. □

The disadvantage of doing it this way is that we do not get any information about the dimension of the variety.

Example (Ex. 7.2(b)). If $C \subset \mathbb{P}^2$ is a curve with $C = Z(f)$, $\deg f = d$, then $P_C(z) = dz + (1-g)$ for an integer $g \in \mathbb{Z}$.

$$0 \to S_{l-d} \xrightarrow{f} S_l \to S/\langle f \rangle_l \to 0$$

so from $\dim_k S_l = \binom{l+2}{2}$ we have

$$\begin{aligned}\phi_C(l) &= \binom{l+2}{2} - \binom{l+2-d}{2} = \frac{1}{2}(l+2)(l+1) - \frac{1}{2}(l-d+2)(l-d+1) \\ &= \frac{1}{2}(l^2 + 3l + 2 - l^2 + 2ld - d^2 - 3l + 3d - 2) = dz - \frac{1}{2}(d^2 - 3d)\end{aligned}$$

so in fact $g = (d-1)(d-2)/2$.

For intersection multiplicities, for curves C, D that *intersect transversally* at P (i.e. they have distinct tangent lines at the point of intersection, where the *tangent line* is the linear part of the curve), then $S/(I(C) + I(D)) = S/\langle f, g \rangle = k[x_0] = 1$, so the intersection multiplicity is 1 as one would expect. This also holds in higher dimension for smooth varieties.

We conclude with:

Example (Classification problem). We would like to classify all subvarieties of $\mathbb{P}^n$ by dimension, degree, and by other numerical invariants. We start with degree $d = 1$.

Claim (Ex. 7.6). If $d = 1$, then Y^r is a linear variety, i.e. $I(Y) = \langle \ell_1, \ldots, \ell_{n-r}$ where the ℓ_i are linear forms.

Proof. If $I(Y)$ is linear, then $S/I(Y) = k[x_0, \ldots, x_r]$ which has Hilbert polynomial $\binom{z}{r}$ which starts $z^r/r! + \ldots$, hence the degree is 1. For the contrary, we argue by induction on the dimension. If the dimension is 0, then we have a single point and hence the degree is 1. In general, let $H \subset \mathbb{P}^n$ be a hyperplane such that $H \not\subset Y$. Suppose $H = Z(x_i)$; Then $H \cap Y$ has degree 1 by (7.7) and induction, so if $I(Y) = \langle f_1, \ldots, f_t \rangle$ then $I(Y \cap H) = \langle f_1, \ldots, f_t, x_i \rangle$ is generated by linear forms. If we repeat with each x_i then we see that the f_i themselves are linear, so Y is linear. □

Claim (Ex. 7.8). If $d = 2$, then Y^r is a quadric hypersurface in $\mathbb{P}^{r+1}$.

Proof. Let $P \in Y^r$ be a fixed point and $Q \neq P \in Y$ a variable point. Consider W, the union of the lines through P and Q contained in $\mathbb{P}^n$. This is a variety of dimension $r+1$. Hence $\deg W \leq \deg Y - 1 = 1$ (counting points of a generic intersection), so $\deg W = 1$ and W is a linear space by the preceding argument. Therefore $Y^r \subset \mathbb{P}^{r+1} \subset \mathbb{P}^n$, and so by the Hauptidealsatz, Y must be generated by a single irreducible polynomial f as claimed. □

§I (Supplement): Representing families (Lines in $\mathbb{P}^3$)

There are many examples of how one can represent families of objects by one algebraic variety:

(1) Conics in $\mathbb{P}^2$ are given by $a_0x_0^2 + a_1x_0x_1 + \cdots + a_5x_2^2 = 0$, so this corresponds to an open set $U \subset \mathbb{P}^5$ since we require that the conic be irreducible.
(2) Linear spaces of dimension r in $\mathbb{P}^n$ are parameterized by the *Grassman variety* $G(r, n)$.
(3) Curves of a fixed genus g are given by a variety of moduli $\mathscr{M}$.
(4) For a fixed curve, the set of divisors classes modulo linear equivalence is the *Jacobian variety*.
(5) For any variety X, we can look at 0-cycles on X modulo equivalence, the *Albanese variety*.
(6) For $V^r \subset \mathbb{P}^n$, the varieties V with a fixed Hilbert polynomial give the *Hilbert scheme*.
(7) Hyperplanes of degree d in $\mathbb{P}^n$ are parameterized by $\mathbb{P}^{\binom{n+d}{n}-1}$.

A line $L \subset \mathbb{P}^3$ is given by two linear equations:

$$a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$$
$$b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$$

This is equivalent to the two-dimensional subspaces of k^4 if we represent these as vectors. Therefore we introduce *Plücker coordinates*:

$$p_{01} = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}, \; p_{02} = \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}, \; \ldots, \; p_{23} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

At least one of these is nonzero (so that the rank of the matrix is 2). Therefore we have points $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$ which depend exactly on the line L.

If we count parameters of the number of lines in $\mathbb{P}^3$, for any point P we have 3 degrees of freedom for lines and the same for a point Q, but along the line L between them the determinants are the same, so there are $3-1$ degrees for P (resp. $3-1$ for Q), totalling 4, so this only a subset of $\mathbb{P}^5$.

Observe that $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$. This quadric Q is a hypersurfaces in $\mathbb{P}^5$, and is nonsingular.

Claim. The set of lines L in $\mathbb{P}^3$ is in bijection with the set of points of Q by the above map ψ.

Proof. Consider an open affine $p_{01} \neq 0$ in Q. By changing coordinates we can assume the matrix with rows a_i and b_j is just

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}$$

and hence $p_{02} = b_2$, $p_{03} = b_3$, $p_{12} = -a_2$, $p_{13} = -a_3$, $p_{23} = a_2b_3 - b_2a_3$. Therefore ψ is injective because b_2, b_3, a_2, a_3 are determined uniquely by $p_{02}, p_{03}, p_{12}, p_{13}$. To see that ψ is surjective, we define the a_2, a_3, b_2, b_3 from the above equations. $\square$

There are also natural subvarieties of Q. For a hyperplane $H \subset \mathbb{P}^3$, the union of lines $L \subset H$ is isomorphic to $\mathbb{P}^2$ and corresponds to a line $\sigma_H \subset Q$. Any two hyperplanes $H \neq H'$ intersect in exactly one line, corresponding to a single point on Q, $\sigma_H \cap \sigma_{H'}$.

If we fix a point $P \in \mathbb{P}^3$, the union of lines $L \ni P$ is again isomorphic to $\mathbb{P}^2$ and corresponds to a line $\sigma_P \subset Q$. If $P' \neq P$, then $\sigma_P \cap \sigma_{P'}$ consists of a single point, the unique line that goes through each of these points.

If we have a hyperplane H and a point P, if $P \notin H$, then $\sigma_P \cap \sigma_H = \emptyset$, whereas if $P \in H$, $\sigma_P \cap \sigma_H = \sigma_{P,H} \simeq \mathbb{P}^1$ $(P \in L \subset H)$. (This situation is similar to a quadric surface in $\mathbb{P}^3$, which is ruled by lines.)

We have:

$$
\begin{tikzcd}
 & Q \arrow[d, dash] & \\
 & \sigma^3_{L_0} \arrow[dl, dash] \arrow[dr, dash] & \\
\sigma^2_H \arrow[dr, dash] & & \sigma^2_P \arrow[dl, dash] \\
 & \sigma^1_{P,H} \arrow[d, dash] & \\
 & \{L_0\} &
\end{tikzcd}
$$

where $\sigma_{L_0} = \{L : L \cap L_0 \neq \emptyset\}$. For if we suppose that $L_0 : x_0 = x_1 = 0$, then the line $L : a_0x_0 + \cdots = b_0x_0 + \cdots = 0$ becomes $a_2x_2 + a_3x_3 = b_2x_2 + b_3x_3 = 0$, so there should be a solution (i.e. a linear dependence), which gives $p_{23} = a_2b_3 - a_3b_2 = 0$. Therefore $\sigma_{L_0} = Q \cap H_{p_{23}}$, the hyperplane in $\mathbb{P}^5$ defined by $p_{23} = 0$. This is the subvariety $p_{02}p_{13} - p_{03}p_{12} = 0 \subset \mathbb{P}^4$, and σ_{L_0} is the cone over this surface, with singularity at exactly $\{L_0\}$.

The natural generalization of this is to linear varieties $L^r \subset \mathbb{P}^n$. This corresponds to a nonsingular projective variety $G(r, n)$ of dimension $(r+1)(n-r)$, and the *Schubert cycles* σ as discussed above live in these varieties. We have flags, which are $P_0 \in L_0 \subset H_0 \subset \ldots$. There are lots of these Schubert cycles and all intersections and so forth are calculable.

§II.1: Sheaves

We have now studied varieties V which are topological spaces defined over an algebraically closed field k with regular functions $\mathscr{O}(U)$ defined on open sets U. Now we forget the ground field k and that the elements of $\mathscr{O}(U)$ are functions; we are left with just a topological space V and on open subsets U an abelian group $\mathscr{O}(U)$, with a "restriction homomorphism" $\mathscr{O}(U) \to \mathscr{O}(W)$ whenever $W \subset U$. This is the motivation for studying sheaves.

Example. Sheaves are in some sense given by "local data." In particular, the presheaf $\mathscr{F}$ given by $\mathscr{F}(X) = \mathbb{Z}$ and $\mathscr{F}(U) = 0$ for all $U \subsetneq X$ is *not* a sheaf.

This shows that (II.1.1) is false for presheaves, since the map from $\mathscr{F}$ to the zero sheaf is an isomorphism on stalks (we ignore $\mathscr{F}(X)$ when computing the direct limit).

The associated sheaf $\mathscr{F}^+$ is in fact the zero sheaf.

Example. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, and if for all $P \in X$, there exists an isomorphism $\mathscr{F}_P \simeq \mathscr{G}_P$, this does *not* imply that $\mathscr{F} \simeq \mathscr{G}$ as sheaves. One must have compatibility of the maps themselves.

As an example, we consider the circle $\mathbb{S}^1$, and the constant sheaf (1.0.3) $\mathscr{Z}$ which has $\mathscr{Z}(U) = \mathbb{Z}$ for all connected open sets U. In particular, the sheaf $\mathscr{Z}$ has global sections $\mathscr{Z}(\mathbb{S}^1) = \mathbb{Z}$. However, if we consider the twisted circle M, obtained by gluing together two open semicircles $U, V \subset \mathbb{S}^1$ with a twist on one side (think of an infinitessimal Möbius strip), and define the constant sheaf on M, then $\mathscr{Z}$ has no global sections (essentially but not rigorously, it is because if one follows a section around the circle on open sets, one will eventually return to the negative of the section at the original point).

Claim. A morphism of presheaves $f : \mathscr{F} \to \mathscr{G}$ is monic if for all U, $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

Proof. In categorical language, f is monic if for all diagrams

$$\begin{tikzcd} \mathscr{X} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \mathscr{F} \arrow[r, "f"] & \mathscr{G} \end{tikzcd}$$

with $f \circ u = f \circ v$ one has $u = v$. If f is injective, then this holds; conversely, if $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is not injective for some U, define a presheaf $\mathscr{Z}_U$ by the data: if $V \not\subset U$ then $\mathscr{Z}_U(V) = 0$, and if $V \subset U$, $\mathscr{Z}_U(V) = \mathbb{Z}$ with the obvious restriction maps. This is a constant presheaf $\mathscr{Z}_U$. We have a map $\mathscr{Z}_U \to \mathscr{F}$ from $\mathbb{Z}$ to any abelian group A by $1 \mapsto a$ for $a \in A$ such that $f(a) = 0$; but we can also take $1 \mapsto 0$, so f is not a monomorphism (in the category of presheaves).

For sheaves, we have a map $\mathscr{Z}_U \to \mathscr{Z}_U^+$ to the associated sheaf, i.e.

$$\begin{tikzcd} \mathscr{Z}_U^+ \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F} \arrow[r, "f"] & \mathscr{G} \\ \mathscr{Z}_U \arrow[u] \arrow[ur] & & \end{tikzcd}$$

so by the above, f is not a monomorphism. $\square$

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, then $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open U iff $f_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective by the first part of (1.1); this is resonant with the idea that sheaves are defined by local data). For the same reason, a morphism of sheaves is an epimorphism iff it is surjective on stalks.

We also have a notion of a kernel of the map $\mathscr{F} \xrightarrow{f} \mathscr{G}$. In general, we may look at an *additive category* $\mathfrak{C}$, i.e. a category with the property that for all objects $X, Y \in \mathfrak{C}$, $\mathrm{Mor}(X, Y)$ is an abelian group such that $\mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \xrightarrow{\circ} \mathrm{Mor}(X, Z)$ is a bilinear map. For such a category, a subobject of an object X is an equivalence class (Y, i) with $Y \xrightarrow{i} X$ with i a monomorphism such that $(Y, i) \equiv (Y', i')$ if there exists an isomorphism

$$\begin{tikzcd} Y \arrow[r, "i"] & X \\ Y' \arrow[u, dotted] \arrow[ur, "i'"] & \end{tikzcd}$$

such that the diagram commutes.

If $\mathfrak{C}$ is an additive category, then we have a notion of kernel of $X \xrightarrow{f} Y$: the kernel $\ker f$ is a subobject (K, i) of X such that $fi = 0$ and for all $Z \xrightarrow{u} X$ such that $fu = 0$, the map factors through K:

$$\begin{tikzcd} K \arrow[r, "i"] & X \arrow[r, "f"] & Y \\ Z \arrow[u, dotted] \arrow[ur, "u"] & & \end{tikzcd}$$

To apply this to sheaves, let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Let $\mathscr{K}(U) = \ker f_U$. We have the commutative diagram

$$\begin{tikzcd} \mathscr{K}(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[d] \\ \mathscr{K}(V) \arrow[r] & \mathscr{F}(V) \end{tikzcd}$$

whenever $V \subset U$. From this it is already clear that $\mathscr{K}$ is a kernel in the category of presheaves.

Claim. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, then $\mathscr{K}$ is a sheaf.

Proof. If $U = \bigcup_i U_i$, then

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ \mathscr{K}(U) \arrow[r] \arrow[d] & \mathscr{K}(U) \arrow[r] \arrow[d] & \mathscr{K}(U) \arrow[d] \\ \mathscr{F}(U) \arrow[r] \arrow[d] & \prod_i \mathscr{F}(U_i) \arrow[r] \arrow[d] & \prod_{i,j} \mathscr{F}(U_i \cap U_j) \arrow[d] \\ \mathscr{G}(U) \arrow[r] & \prod_i \mathscr{G}(U_i) \arrow[r] & \prod_{i,j} \mathscr{G}(U_i \cap U_j) \end{tikzcd}$$

Now apply the snake lemma. □

We also have cokernels (if we reverse arrows), for which we will need a notion of epimorphism. We would like to say that f is epic iff it is surjective on open sets. After all, if $\mathscr{F}(U) \to \mathscr{G}(U) \to C \to 0$ is not surjective for some U, we define the constant presheaf $\mathscr{Z}$ as before which gives proves that f is not epic. However, for the associated sheaf $\mathscr{Z}^+$ there may not be a map $\mathscr{G} \to \mathscr{Z}^+$.

In particular: for sheaves, $f : \mathscr{F} \to \mathscr{G}$ an epimorphism $\not\Leftrightarrow$ $\mathscr{F}(U) \to \mathscr{G}(U)$ is surjective. However, as it was remarked above, we do have that $\mathscr{F} \to \mathscr{G}$ is an epimorphism iff for all $P \in X$, $\mathscr{F}_P \to \mathscr{G}_P$ is surjective. (For the converse, we need that $\mathscr{H}(U) = \mathscr{G}(U)/\operatorname{img}\mathscr{F}(U)$ has

$$\mathscr{F} \to \mathscr{G} \to \mathscr{H}^+ \to 0.$$

Example. Here is an example of this phenomenon. Let $X = \mathbb{A}^1$, and take two points $P \neq Q \in \mathbb{A}^1$. Let $\mathscr{F} = \mathbb{Z}$ be the constant sheaf on $\mathbb{Z}$, and let $\mathscr{G} = \mathscr{Z}_P \oplus \mathscr{Z}_Q$ where $\mathscr{Z}_P(U) = \mathbb{Z}$ if $P \in U$ and $\mathscr{Z}_P(U) = 0$ otherwise. Then $\mathscr{F} \to \mathscr{G}$ is an epimorphism of sheaves which is not surjective on open sets.

§II.2: Schemes

To define the structure sheaf of rings $\mathscr{O} = \mathscr{O}_X$ on an affine scheme $X = \operatorname{Spec} A$, one can also use the following construction: for each open $U \subset X$, define a presheaf on the base $\{D(f)\}$ where $D(f) = X \setminus V(f)$ by $\mathscr{O}_X(D(f)) = A_f = A[1/f]$ (note this localization is the collection of elements a/f^r where $a_1/f^{r_1} = a_2/f^{r_2}$ if there exists s such that $f^s(f^{r_2}a_1 - f^{r_1}a_2) = 0$; the kernel of the localization consists of those elements annihilated by f). Note that if $D(g) \subset D(f)$, then $V(g) \supset V(f)$, and hence $g \in \sqrt{\langle f \rangle}$, so $g = f^r h$, and $1/g = 1/f^r h$ and thus we get a well-defined restriction map $A_f \to A_g$. Now take the associated sheaf; it will have the same stalks $\mathscr{O}_P$ as the presheaf. Although the sections could in theory now be anything, we find:

Proposition (Proposition 2.2)**.** *If A is a ring, $X = \operatorname{Spec} A$, $\mathscr{O}_X$ the structure sheaf, then $\mathscr{O}_P = A_\mathfrak{p}$, $\mathscr{O}_X(D(f)) = A_f$, and $\mathscr{O}_X(X) = A$.*

Ideas of proof. We know that $\mathscr{O}_P = \varinjlim_{D(f) \ni P} A_f$ is the stalk of the presheaf. Now $D(f) \ni P = \mathfrak{p}$ iff $f \not{i}n\mathfrak{p}$, so this is just $A_\mathfrak{p} = \{a/s : s \notin \mathfrak{p}\}$.

To see the second statement, we note that if $f \in A$, then $D(f) = \operatorname{Spec} A_f \subset X = \operatorname{Spec} A$ has the induced topology, so we can replace A by A_f and prove the latter. If we have $\mathscr{O}_X(X) \xleftarrow{\phi} A$, we will show ϕ is injective. If $a \in A$, $\phi(a) = 0$ in the sheaf, the there exist $f_1, \dots, f_n$ such that the $D(f_i)$ cover X and $a|_{D(f_i)} = 0$. The $D(f_i)$ cover X iff $\langle f_i \rangle_i = \langle 1 \rangle$, iff $1 = \sum_{i=1}^n a_i f_i$ (which shows that $\operatorname{Spec} A$ is quasi-compact—every open cover has a finite subcover). If $a = 0 \in A_f$ then there exists n_i such that $f_i^{n_i} a = 0$, so let $N = n \max_i n_i + 1$. Then $1 = \langle f_i \rangle_i = \langle f_i \rangle_i^N$, so $1 = \sum_I b_I f_I^{r_I}$ with $\sum_i r_i = N$, and thus $a = \sum_I b_I f^I a = 0$. Thus ϕ is injective.

To show ϕ is surjective is harder: to show that $\alpha \in \mathscr{O}_X(X)$ came from $a_i \in A_{f_i}$ with the $D(f_i)$ covering X one must show $a_i|_{D(f_i f_j)} = a_j|_{D(f_i f_j)}$. □

Example. The scheme $\operatorname{Spec} \mathbb{Z}$ consists of a line of primes $\langle 2 \rangle, \langle 3 \rangle, \dots$ as well as a generic "fuzzy" point $\langle 0 \rangle = \zeta$ whose closure is the entire space. An open set U leaves out a finite set of finite primes p_i, and then $\mathscr{O}(U) = \mathbb{Z}[1/p_i]_i$.

Example. If $k = \mathbb{F}_2$ is a finite field, $\operatorname{Spec} k[x] = \operatorname{Spec} \mathbb{F}_2[x]$ is still dimension 1 (we retain the topological definition of dimension, and in the new topology $\dim \mathbb{A}_k^n = n$ for all fields k). The space consists of points $\langle f(x) \rangle$ for polynomials $f(x) \in \mathbb{F}_2[x]$ irreducible over $\mathbb{F}_2$, as well as a generic point $\langle 0 \rangle$ whose closure again is the entire space.

Example. If we let $A = k[\epsilon]/\langle \epsilon^2 \rangle$ (called the *ring of dual numbers*), then $\operatorname{Spec} A$ consists of a single point (the only prime ideal is $\langle \epsilon \rangle$), but it has an "infinitesimal arrow sticking out of the point" which represents the nilpotent.

Example. The scheme $\operatorname{Spec}\mathbb{Z}[x]$ is two-dimensional, with $\langle p, f(x)\rangle$ closed points for polynomials $f(x) \in \mathbb{Z}[x]$ which are irreducible modulo p. There are non-closed points $\langle p\rangle$ for p prime and $\langle f(x)\rangle$ for $f(x)$ irreducible, whose closure is the union of all closed points above which contain it. There is also the generic point $\langle 0\rangle$.

One should compare the definition of an affine scheme with the definition of a complex manifold M where each open set $U_i \subset M$ is isomorphic to a $\Delta_i \subset \mathbb{C}^n$ with agreement on $U_i \cap U_j$. For the scheme, the sheaf "contains" the compatibility information.

Example. Here is an extended example of gluing: if $X = Y = \mathbb{A}^1$, $U = X \setminus \{0\}$, $V = Y \setminus \{0\}$, then gluing using the identity morphism gives the "affine line with a doubled origin".

If we now glue using the map $x \mapsto 1/x$, we obtain an origin and a point at ∞; they glue to give the projective line $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$.

In order to explain (2.6), we note that to any variety V over an algebraically closed field k, which for simplicity we assume has $V \subset \mathbb{A}^n_k$ given by an ideal $I \subset A$, we have an associated scheme V^{sch}, which as a set consists of the closed irreducible subsets of V (containing all of the points of V which are just the closed points of V^{sch}; note that this space has the same open sets, and is just the scheme $\operatorname{Spec}(A/I)$.

For the converse, in order to associate a variety to a given scheme we need the following: first, it must be defined over an algebraically closed field, which is to say we have a map $X \to \operatorname{Spec} k$; second, we need it to be *quasiprojective*, which is to say we need an injective map $X \hookrightarrow \mathbb{P}^n_k$; finally, we need X to be *integral*, which is to say we need the rings $\mathscr{O}(U)$ to be domains for all U (in order that the scheme be irreducible). Note (for the experts) that the quasiprojective condition contains within it that the scheme is separated and of finite type over k.

§II.3: First Properties of Schemes

There are certain properties of a scheme X which only depend on the topology of the underlying space, which we denote $\operatorname{sp}(X)$, for example *connected*, *irreducible* ($X \neq Y \cup Z$ if Y and Z are proper closed subsets), *quasi-compact* (every open cover has a finite subcover), *noetherian* (as a topological space, this is the descending chain for closed subsets). Note that noetherian is equivalent to every open subset is quasi-compact, and noetherian implies quasicompact. We also have the notion of *dimension*, which is $\sup_n\{n : Z_0 \subsetneq \cdots \subsetneq Z_n \subset Y\}$ for Z_i closed irreducible subsets.

There are also properties that depend on the scheme structure itself. We have notions of *reduced* ($\mathscr{O}_X(U)$ has no nilpotents for all open U), *integral* ($\mathscr{O}_X(U)$ is a domain for all U). Being reduced is a local condition (it is equivalent to demand for all $P \in X$, $\mathscr{O}_{P,X}$ has no nilpotents) but being integral is not: $\mathscr{O}_{P,X}$ integral for all P does not imply that $\mathscr{O}_X(U)$ is a domain, take for example k a field, $X = \mathbb{A}^1_k \cup \mathbb{A}^1_k$, we have $\mathscr{O}_X(X) = k[x] \times k[x]$ not a domain even though around every point $\mathscr{O}_{P,X} = k[x]_{\mathfrak{m}_P}$.

Here is another proof of (3.1) that integral is equivalent to reduced and irreducible:

Proof. Integral implies reduced and if $X = Y \cup Z$, then $Y \cap Z$ is closed, $X \setminus (Y \cap Z) = U \sqcup V$ for (disjoint) open sets U and V, and so on this open set the corresponding ring cannot be integral.

If X is reduced and irreducible, cover X with open affines U; $\mathscr{O}_X(U)$ has no nilpotent elements, so if $ab = 0$, then $\langle 0\rangle = \langle ab\rangle \subset \langle a\rangle \cap \langle b\rangle$, and we have equality because if $x = ay = bz$, then $x^2 = abyz = 0$ so $x = 0$, so $U = V(a) \cup V(b)$ and this will lift to a decomposition of X, a contradiction. $\square$

Example. Here is an example for (3.1.1): The fact that $\operatorname{sp} X$ is a noetherian topological space does not imply that $(X, \mathscr{O}_X)$ a noetherian scheme. Take $X = \operatorname{Spec} A$ for $A = k[x_1, x_2, \dots]/\langle x_1^2, x_2^2, \dots\rangle$. A is clearly not noetherian, but $\operatorname{Spec} A$ consists of the sole prime ideal $\langle x_1, x_2, \dots\rangle$ (since any prime must contain $0 = x_i^2$ and hence each x_i and no other translate). This is a point with "infinitely many infinitesimal arrows", but a point itself is certainly noetherian.

Example. The scheme which is an infinite union of affine lines over a single affine line is locally of finite type but itself is not of finite type.

Example. If B is any ring, $\mathbb{A}^n_B = \operatorname{Spec} B[x_1, \dots, x_n]$ has a map $\mathbb{A}^n_B \to \operatorname{Spec} B$ induced by $B \hookrightarrow B[x_1, \dots, x_n]$ which is of finite type by definition (even if B is a very nasty ring).

Example. If $X \to Y = \operatorname{Spec} k$ is a finite map, then by definition $X = X_1 \sqcup \cdots \sqcup X_r$ where $X_i = \operatorname{Spec} k_i$ and $[k_i : k] < \infty$.

Example. The map $\operatorname{Spec} k[x, y]/\langle x - y^2\rangle \to \operatorname{Spec} k[x] = \mathbb{A}^1_k$ is finite because the ring $k[x, y]/\langle x - y^2\rangle$ is a finite module over $k[x]$ (it is generated as a module by $1, y$).

The definition of an open subscheme should be refined as follows:

Definition. A morphism $f : X \to Y$ is an *open immersion* if $f(X) = V \subset Y$ is an open subset and $f : X \to V$ induces an isomorphism of the image with the induced scheme structure.

An *open subscheme* is an equivalence class (X, f) with equivalence if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "i"] & Y \\ X' \arrow[ur, "f'"] & \end{tikzcd}$$

where i is an isomorphism.

The reason we insist on the surjectivity of $\mathscr{O}_X \to f_*\mathscr{O}_Y$ is because if $k_1 \subset k_2$ is any field extension, we would have a morphism $\operatorname{Spec} k_2 \to \operatorname{Spec} k_1$, but we would not want to think of $\operatorname{Spec} k_2$ as a closed subscheme. Note that it is equivalent to require that the map $\mathscr{O}_{f(P),Y} \to \mathscr{O}_{P,X}$ is surjective for all points $P \in X$.

Example. If $k = \overline{k}$, $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$, we wish to look for all possible closed subschemes. Topologically, Y must either be X or a finite set of points. Suppose we have a map $Y \xrightarrow{f} X$ which makes Y into a nonproper closed subscheme, i.e. $f(Y) = X$ and $\mathscr{O}_X \xrightarrow{f^\sharp} f_*\mathscr{O}_Y$ surjective. The claim is that $f^\sharp$ is also injective, for if $U \subset X$ is an open set, and $s \in \mathscr{O}_X(U)$ such that $f^\sharp_U s = 0$ in Y, if $s \neq 0$ then we can define $Z(S) = \{P \in X : s(P) \in \mathfrak{m}_P \subset \mathscr{O}_P\}$. This subset is proper and closed, and $f^{-1}(Z(s)) = Z(f^\sharp s)$, so $Y \subset Z(s)$, a contradiction.

Therefore the closed subschemes are exactly finite unions of points. If we take just the origin, then we have the prime $\langle 0\rangle$, corresponding to the ideal $I = \langle x\rangle \subset k[x]$, and we have the closed subscheme $Y = \operatorname{Spec} k[x]/\langle x\rangle = k$, which is what one would expect, a simple point. But we can take any ideal I with $Z(I) = \{0\}$, namely

$I \subset \langle x \rangle$, so $\langle x \rangle = \sqrt{I}$, and thus $x^n \in I$ and so $I = \langle x^n \rangle$ (since $k[x]$ is a PID). The subscheme $Y = \operatorname{Spec} k[x]/\langle x^n \rangle$ is a point with an "infinitesimal tangent direction". (The corresponding ring is an Artin ring; it has finite length which as a vector space over k has a basis $1, x, \dots, x^{n-1}$.)

In general, if X is affine, $X = \operatorname{Spec} A$, and $Y \xrightarrow{f} X$ is a closed immersion, then Y is affine and $Y = \operatorname{Spec} A/I$ for some ideal $I \subset A$. We will see this later.

Example. If we take $\mathbb{A}^2_k$ and the closed subscheme $I = \langle x, y \rangle \cap \langle y - x^2 \rangle = \langle xy - x^3, y^2 - x^2 y \rangle$, then this is a parabola with an *embedded point* at the origin. It has the nilpotent $f^2 = (y - x^2) = yf - x^2 f = 0$. The prime $\langle f \rangle$ is a minimal prime, and the prime $\langle x, y \rangle$ is an embedded prime.

It is useful now to extend the definition of the Hilbert polynomial to any closed subscheme of projective space:

Definition. If $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n] = \operatorname{Proj} S$ and $I \subset S$ is homogeneous, then $Y = \operatorname{Proj} S/I \to \operatorname{Proj} S = X$ is a closed immersion. We define the *Hilbert polynomial of* Y $P_Y(z)$ as the polynomial which agrees with $\phi_Y(z) = \operatorname{len}(S/I)_z$ for z sufficiently large. We have as before that the degree Y is the leading coefficient times the factorial of the dimension of Y.

Using this, we can investigate:

Example. The curves (dimension 1) subschemes of degree 2 contained in $\mathbb{P}^2_k$, e.g. $yz - x^2 = 0$ can be categorized first if they have no embedded component (i.e. if in each open affine $U = \operatorname{Spec} A$, I the ideal defining Y, I has no embedded primes, which since $I \subset k[x, y]$ is height 1 and $k[x, y]$ is a UFD, implies that I is principal by the Hauptidealsatz). In this case, f is a degree 2 form, and is either irreducible (defining a conic), breaks up as the product of two distinct linear forms (and is the union of two lines) or is the square of a linear factor (and is an infinitesimal double line, with nilpotents at every point of the line).

Remark. In the proof of (3.3), it is important to distinguish the cases when the intersection of two affines is affine. This is not true in general: we can take the affine plane with a doubled origin, for example. If Y is affine, then the intersection of two open affines is also affine (look at the diagonal $Y \xrightarrow{\Delta} Y \times Y$—the subscheme $\Delta(Y) \cap (Y_1 \times Y_2)$ is affine and is isomorphic to $Y_1 \cap Y_2$), and we will see is also true in general for separated schemes.

Here is a proof that if we have the fibre product $X_s = X \times_S \operatorname{Spec} k$, where $k = k(s) = \mathscr{O}_{s,S}/\mathfrak{m}_s$ for some $s \in S$, i.e.

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

then $\frac{X}{s}$ is homeomorphic to $f^{-1}(s) \subset X$. We may assume S is affine (since restricting to an affine in which s sits will preserve the fibre product by its universal property), and then assume that X is affine (since we may look at the union of affine opens, as verifying a map is a homeomorphism is local on the image). We then have the dual diagram:

$$\begin{tikzcd} A \otimes_R k(\mathfrak{p}) = A/\mathfrak{p} & A \arrow[l] \\ k(\mathfrak{p}) \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

where $S = \operatorname{Spec} S$ and $X = \operatorname{Spec} A$, s corresponds to the prime ideal $\mathfrak{p}$. Then the map $R \to A$ is induced by the map which takes $\mathfrak{p}$ to the set of ideals $\mathfrak{q} \in A$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. But this is exactly the map on the left, since $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq A/\mathfrak{p}$. (Localize for more precise information.)

Here is a final example of a family of schemes:

Example. Take the map $X = \mathbb{A}^2 \to S = \mathbb{A}^1$ where $S = \operatorname{Spec} k[t]$ and $X = \operatorname{Spec} S[x] = \operatorname{Spec} k[t][x]$, where $\operatorname{char} k \neq 2$. Restrict it to a map $Y = \operatorname{Spec} k[t][x]/\langle t - x^2\rangle$ to S induced again by the inclusion $k[t] \hookrightarrow k[t][x]/\langle t - x^2\rangle$. Then if $t = 1$, Y_1 consists of two distinct points; if $t = 0$, we have $Y_0 = \operatorname{Spec} k[x]/\langle x^2\rangle$, a double point; and for the generic point $t = \zeta$, $Y_\zeta = \operatorname{Spec} k(t)[x]/\langle t - x^2\rangle$ which comes from a degree two extension $k(y) \to k(x)$ of the residue field. Note that in each case, the fiber is length 2 over its base.

§II.4: Separated and Proper Morphisms

Here are some examples of separated schemes.

Example. If $k = \overline{k}$, then $\mathbb{P}^n_k$ is separated. We have

$$\begin{tikzcd} \mathbb{P}^n_k \arrow[r, "\Delta"] \arrow[d] & \mathbb{P}^n_k \times_k \mathbb{P}^n_k \arrow[r] \arrow[dl] & \mathbb{P}^N_k \\ \operatorname{Spec} k & & \end{tikzcd}$$

by the Segre embedding. But if we instead view the product as given by bihomogeneous polynomials in $k[x_0, \dots, x_n; y_0, \dots, y_n]$, then the image of the diagonal is given by the ideal $x_i y_j - x_j y_i$ for i, j (these equations say $x_i/x_j = y_i/y_j$ wherever these are defined).

Example. If $f : X \to S$ is separated, and $U \subset X$, then the map $f|_U : U \to S$ is separated, as $\Delta(U) \cap U \times_S U$ is closed in $U \times_S U$, an open subset of $X \times_S X$. Similarly, if $Y \subset X$ is closed, then $Y \times_S Y \hookrightarrow X \times_S X$ is closed so $Y \to S$ is separated.

Example. It is a general fact that if $Y \subset X = \operatorname{Spec} A$ is a closed subscheme of an affine scheme, then $Y \simeq \operatorname{Spec} A/I$ for some ideal I. In this case $\Delta(X) \cap Y \times_S Y = \Delta(Y)$.

Therefore if $k = \overline{k}$, and V/k is a variety, then the associated scheme V^{sch} over $\operatorname{Spec} k$ is separated, because it is either (quasi-)affine or (quasi-)projective.

Topologically, a space X is Hausdorff iff $\Delta(X) \subset X \times X$ is closed: Take any two points P and Q, and take a neighborhood around (P, Q) which does not meet Δ (since $\Delta(X)$ is closed); this gives neighborhoods $U \ni P$ and $V \ni Q$ such that $U \cap V = \emptyset$. This is *not* the same thing in algebraic geometry, because ${}^{(}X \times_S X) \neq{}^{(} X) \times{}^{(} X)$. For example, $\mathbb{A}^1 \times_k \mathbb{A}^1 = \mathbb{A}^2 \neq \mathbb{A}^1 \times \mathbb{A}^1$, where on the right we take the topological product (for example, $\mathbb{A}^2$ contains curves). So in algebraic geometry the space is not the product of the spaces, and the topology is not the product of the topologies!

Here is the motivation for the valuative criterion of separatedness. Let X be a curve over k and consider the function field $K = K(X)$. We have seen that there is a unique nonsingular projective curve $\widetilde{X}$ birational to $X \supset X_{\text{reg}} \hookrightarrow \widetilde{X}$, and X is isomorphic to an open set U of $\widetilde{X}$. If we embed $X \subset \overline{X} \hookrightarrow \mathbb{P}^n$, then we get a map $U \to \overline{X}$, and by the pasting points lemma we obtain a map $\widetilde{X} \to \overline{X}$. What we have shown is that for each valuation ring R of K over k, there exists a unique point $P \in \overline{X}$ such that $\mathscr{O}_{P,\overline{X}} \subset R$ is dominated by R. The point is that if X is separated, then each R dominates *at most* one point of X.

Now in terms of proper morphisms, we have $\widetilde{X}$ and $\overline{X}$ over k, where each local ring is dominated by at most one point, but by properness each R dominates at least one point, which is how uniqueness is obtained. As a general statement, "giving a valuation ring is like giving a sequence, and giving a point is like giving a limit."

The valuative criterion of properness is the statement that "every valuation has a center", which means the following: If X is a variety over $\operatorname{Spec} k$ with function field K containing a valuation ring R, then X is proper over k iff for all R, R has a center in X, which is a a point $x \in X$ such that we have a map $\operatorname{Spec} R \to X$, i.e. $\mathscr{O}_{x,X} \to R$ has $\mathfrak{m}_x = \mathfrak{m}_R \cap \mathscr{O}_{x,X}$.

Here is an overview sketch of (4.9):

Theorem. *Every projective morphism $X \to Y$ is proper.*

Sketch of proof. Such a map is separated because $\mathbb{P}^n_Y \to Y$ is separated and $X \hookrightarrow \mathbb{P}^n_Y$ is a closed immersion which is separated. Such a map is of finite type for the same two reasons.

Therefore it is enough to show that $X \to Y$ is universally closed. We have:

$$\begin{tikzcd}
 & \mathbb{P}^m_{Y'} \arrow[ddl, bend left] & \\
X' \arrow[ur] \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \mathbb{P}^n_Y \arrow[dl] \\
Y' \arrow[r] & Y &
\end{tikzcd}$$

This shows that the base change of a projective morphism is projective, so it suffices to show that any projective morphism is closed, for which it is enough to show that $\mathbb{P}^n_Y \to Y$ is closed for any scheme Y. It is enough to show this for $\mathbb{P}^n_{U_i} \to U_i$ for U_i affine, since $f(Z)$ is closed in Y iff $f(Z) \cap U$ is closed in U for all U. So let $U = \operatorname{Spec} A$, and then we have $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n] \to \operatorname{Spec} A$. Let $\mathbb{Z} \subset \mathbb{P}^n_A$ be a closed subset. Z is a finite union of closed irreducibles (by the noetherian hypothesis), so we may assume $f(Z) \subset Y = \overline{f(Z)} \subset \operatorname{Spec} A$ is irreducible, so it corresponds to a prime ideal $\mathfrak{p} \subset A$. If we base extend Y to $\operatorname{Spec} A/\mathfrak{p}$, we obtain

$$\begin{tikzcd}
\mathbb{P}^n_Y \arrow[r] \arrow[d] & Y = \operatorname{Spec} A/\mathfrak{p} \\
Z \arrow[ur, "f"] &
\end{tikzcd}$$

where $\operatorname{Spec} A/\mathfrak{p}$ is integral and $f : Z \to Y$ is dominant, so it is enough to show that $f : Z \to Y$ is surjective.

Let K be the function field of Y, and let Z have the reduced induced structure so that its function field L is an extension of K. Let $y \in Y$ so that $\mathscr{O}_{y,Y} \subset K$. There exists a valuation ring R of L dominating $\mathscr{O}_{y,Y}$. We want to show that R has a center on Z. We may assume Z is not contained in any hyperplane $x_i = 0$ of $\mathbb{P}^n_Y$ (else we could construct a map to a smaller projective space). Thus $x_i/x_j|_Z = f_{ij} \in L$. Let v be the valuation on R, and let $v(f_{i0}) = g_i \in G$, the value group. Choose r such that g_r is minimal over g_i. Then $v(f_{ir}) = g_i - g_r \geq 0$, so $f_{ir} \in R$ for given r and all R. Thus $A[x_0/x_r, \ldots, x_n/x_r] \to R$ gives a map on $U_r = D_+(x_r)$ which is $\operatorname{Spec} R \to Z \subset \mathbb{P}^n$. □

And now a vista on schemes over $\mathbb{C}$ to motivate the theory of properness. Fix $k = \mathbb{C}$, and X a scheme of finite type over $\mathbb{C}$; then we have open sets $U = U_i = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/I \subset X$, with $I = \langle f_1, \ldots, f_q \rangle$. Let $\mathbb{C}^n$ have the usual (Archimedean) topology. Then the elements $f_1, \ldots, f_q$ are holomorphic functions (being polynomials), and $Z \subset \mathbb{C}^n$ by $Z = \{z \in \mathbb{C}^n : f_1(z) = \cdots = f_q(z) = 0\}$ is closed in the usual topology. Define the sheaf $\mathscr{O}_{\mathbb{C}^n}$ as the sheaf of germs of holomorphic functions, thus $\mathscr{O}_Z = \mathscr{O}_{\mathbb{C}^n}/\langle f_1, \ldots, f_q \rangle$, and $(Z, \mathscr{O}_Z)$ becomes a locally ringed space. If we now set Z to be the set of closed points of $U = U_i$, the map $Z \to U$ is a continuous map, bijective on the set of closed points. By gluing, we obtain $(X^{\mathrm{an}}, \mathscr{O}_{X^{\mathrm{an}}})$, the *associated analytic space* to X. An *analytic space* is a locally ringed space $(Y, \mathscr{O}_Y)$ covered by open sets U_i such that $(U_i, \mathscr{O}_X|_{U_i}) \simeq Z_i \subset D \subset \mathbb{C}^n$, where the Z_i are zero sets of functions f_i and $\mathscr{O}_Z = \mathscr{O}_D/\langle f_i \rangle$.

In sum, given a scheme X of finite type over $\mathbb{C}$, we obtain X^{an} over $\mathbb{C}$, an analytic space, with a continuous map $X^{\mathrm{an}} \to X$.

Here are facts about this construction:

- X is separated iff X^{an} is Hausdorff.
- X is reduced iff X^{an} is reduced.
- X is connected iff X^{an} is connected.
- X is nonsingular (X is integral and for all $x \in X$, $\mathscr{O}_{x,X}$ is regular) iff X^{an} is a $\mathbb{C}$-manifold.
- And most relevant, X is proper over $\mathbb{C}$ iff X^{an} is compact.

Therefore to X a nonsingular proper algebraic variety over $\mathbb{C}$, we obtain X^{an} a compact $\mathbb{C}$-manifold. Is there a converse? Well, if $\mathcal{X}$ is a compact Riemann surface of dimension 1, then it is a fact that it is given in this way by a nonsingular algebraic curve X so that $X^{\mathrm{an}} = \mathcal{X}$. In dimension ≥ 2, the field of global meromorphic functions on X has transcendence degree equal to the dimension of X, but there exist complex compact manifolds with this transcendence degree not equal to its dimension. But we may consider *Moishezon* manifolds, which are compact $\mathbb{C}$-manifolds with equality. In dimension ≥ 3, there exist Moishezon manifolds that are not algebraic, so we have the set of nonsingular projective varieties properly contained in the set of nonsingular proper varieties properly contained in Moishezon manifolds properly contained in the set of compact $\mathbb{C}$-manifolds.

§II.5: Sheaves of Modules

Here is an example to illustrate that the tensor product of two sheaves is not always a sheaf:

Example. Let $X = \mathbb{P}^1_k$, $\mathscr{F} = \mathscr{O}_X(1)$, $\mathscr{G} = \mathscr{O}_X(-1)$. Then $\mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{G}(X) = k^2 \otimes_k 0 = 0$, but $\mathscr{F} \otimes \mathscr{G} = \mathscr{O}_X$ so that $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})(X) = k$.

A word about invertible sheaves: given a sheaf $\mathscr{L}$ which is locally free of rank 1, we have $\mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$ which is another invertible sheaf, and $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^\vee \simeq \mathscr{O}_X$, by the map $s \otimes f \mapsto f(s)$ on an open set. $\operatorname{Pic} X$ is the set of invertible sheaves on X up to isomorphism, and is an abelian group under $\otimes$ with identity $\mathscr{O}_X$. In fact, the category of $\mathscr{O}_X$-modules has also the operation $\oplus$ and so is almost a ring (but there are no additive inverses). Instead, one may define the Grothendieck group of the category by taking the free abelian group on the objects which are $\mathscr{O}_X$-modules and identify $\mathscr{F} - \mathscr{F}' - \mathscr{F}'' = 0$ whenever $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence. This has a ring structure, where again multiplication is given by $\otimes$.

Here are the details of the definition of ϕ_*: If $\phi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, $\mathscr{G}$ a sheaf of $\mathscr{O}_Y$-modules, we have $\phi^{-1}\mathscr{G}$ a sheaf of abelian groups on X by $(\phi^{-1}\mathscr{G})(U) = \varinjlim_{V \supset \phi(U)} \mathscr{G}(V)$. $\phi^{-1}\mathscr{G}$ is a sheaf of $\phi^{-1}\mathscr{O}_Y$-modules on Y: but we also have a map $\phi^\sharp : \mathscr{O}_Y \to \phi_*\mathscr{O}_X$, and therefore we have a map

$$\phi^{-1}\mathscr{O}_Y \to \phi^{-1}\phi_*\mathscr{O}_X \xrightarrow{\theta} \mathscr{O}_X$$

by

$$(\phi^{-1}\phi_*\mathscr{O}_X)(U) = \varinjlim_{V \supset \phi(U)} (\phi_*\mathscr{O}_X)(U) = \varinjlim_{\phi^{-1}(V) \supset U} \mathscr{O}_X(\phi^{-1}(V)) \xrightarrow{\rho} \mathscr{O}_X(U).$$

Hence we define $\phi^*\mathscr{G} = \phi^{-1}\mathscr{G} \otimes_{\phi^{-1}\mathscr{O}_Y} \mathscr{O}_X$.

For inclusions, we have a very different story if $U \subset X$ is open and if $Y \subset X$ is closed. In the first case, $\mathscr{O}_U = \mathscr{O}_X|_U$ makes $(U, \mathscr{O}_U)$ into a ringed space, and if $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules, $\mathscr{F}|_U$ is a sheaf of $\mathscr{O}_U$-modules; if $i : U \to X$ is the inclusion, then $\mathscr{F}|_U = i^*\mathscr{F} = i^{-1}\mathscr{F}$ (since $i(V)$ is open and the tensor product has no effect). Conversely, if $\mathscr{F}$ is a sheaf of $\mathscr{O}_U$-modules, $i_*\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules by $(i_*\mathscr{F})(V) = \mathscr{F}(V \cap U)$. Another method is "extending by zero," obtained from

$$V \mapsto \begin{cases} \mathscr{F}(V), & V \subset U; \\ 0, & \text{else.} \end{cases}$$

The associated sheaf of this presheaf, denoted $i_!\mathscr{F}$, has if $P \in U$ that $(i_!\mathscr{F})_P = \mathscr{F}_P$ and if $P \notin U$, then $(i_!\mathscr{F})_P = 0$. Compare this with $(i_*\mathscr{F})_P = \mathscr{F}_P$ but $(i_*\mathscr{F})_P$ is something interesting.

Example. Take $X = \mathbb{A}^1_k$, $U = X \setminus \{P\}$, and $\mathscr{F} = \mathscr{O}_U$. $i_!\mathscr{F}$ is a sheaf on X which looks like $\mathscr{O}_U$ except at P, where $i_*(\mathscr{F})_P = \varinjlim_{V \ni P} \mathscr{F}(V)$ which are functions with a (finite) pole at 0, so this is actually $k(x)$ by restriction any germ to an open set where its only pole is at zero. Elsewhere, $\mathscr{O}_Q = k[x]_Q$.

Now we look at affine schemes, $X = \operatorname{Spec} A$. If $\mathscr{F}$ is any sheaf, one obtains an $\mathscr{O}_X(X) = A$-module by $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$. This is a functor, because a map $\mathscr{F} \to \mathscr{G}$ gives $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G})$, and in fact an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$$

gives an exact sequence of modules

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H}).$$

This functor is not necessarily right exact, however; we have

$$0 \to \mathscr{I}_{\{P,Q\}} \to \mathscr{O}_X \to k(P) \oplus k(Q) \to 0$$

which gives

$$0 \to \Gamma(X, \mathscr{I}) = 0 \to \Gamma(X, \mathscr{O}_X) = k \to \Gamma(X, k(P) \oplus k(Q)) = k \oplus k \not\to 0.$$

Given A, and M an A-module, we construct $\widetilde{M}$, a sheaf of $\mathscr{O}_X$-modules on $X = \operatorname{Spec} A$. If $f \in A$, $D(f) = \operatorname{Spec} A_f \subset X$ is open and subsets of this form give a base for the topology. We define a presheaf on the base by $D(f) \mapsto M_f$, since if $D(g) \subset D(f)$, we have a map $M_f \to M_g$. We let $\widetilde{M}$ be the associated sheaf.

Proposition (Proposition 5.1)**.** *Let $X = \operatorname{Spec} A$, M, $\widetilde{M}$ be as above.*

(a) $(\widetilde{M})_P = M_{\mathfrak{p}}$;
(b) $\Gamma(D(f), \widetilde{M}) = M_f$;
(c) $\Gamma(X, \widetilde{M}) = M$.

Proof. We have the presheaf stalk

$$(\widetilde{M})_P = \varinjlim_{U \ni P} \widetilde{M}(U) = \varinjlim_{D(f) \ni P} M_f = \varinjlim_{f \notin \mathfrak{p}} M_f = M_{\mathfrak{p}}.$$

This proves (a).

For (b), we note that if we have $D(f) \subset X$, and an A-module M, then we obtain a natural A_f-module M_f and $\widetilde{M_f} = \widetilde{M}|_{D(f)}$. Since the $\widetilde{\ }$ functor can be restricted to open sets contained in $D(f)$, we can prove (c) applied to A_f and M_f.

To prove (c), we have a map $\alpha : \Gamma(X, \widetilde{M}^{\text{pre}}) \to \Gamma(X, \widetilde{M})$. We check that α is injective: suppose $\alpha(m) = 0 \in \Gamma(X, \widetilde{M})$, so $\alpha(m) = 0 \in M_{\mathfrak{p}}$ for all $\mathfrak{p}$ so m is in the kernel fo the map $M \to M_{\mathfrak{p}}$, so there exists an $s \notin \mathfrak{p}$ such that $sm = 0$. The annihilator $\operatorname{Ann}(m) = \{a \in A : am = 0\}$ is an ideal I of A: but for every $\mathfrak{p}$, there exists an $s \notin \mathfrak{p}$ such that $s \in \operatorname{Ann}(m)$, so $I \not\subset \mathfrak{p}$, hence $I = A$, and $1 \in I$, so $1m = m = 0$.

Finally, we check that α is surjective: given a global section of $\widetilde{M}$, there exists a cover of X of the form $\{D(f_i)\}$ (finite because X is quasicompact) and on each $D(f_i)$, $m_i/f_i^{n_i} \in \widetilde{M}_{f_i}^{\text{pre}} = M_{f_i}$ such that on $D(f_i) \cap D(f_j) = D(f_i f_j)$ they agree, which means there exists N_{ij} such that $(f_i f_j)^{N_{ij}}(f_j^{n_j} m_i - f_i^{n_i} m_j) = 0 \in M$. But since

$$\frac{m_i}{f^{n_i}} = \frac{f_i^{n-n_i} m_i}{f_i^n} = \frac{m_i'}{f_i^n}$$

we may make $n = n_i$ for all i, and we can replace N with the maximum of the N_{ij}. From $(f_i f_j)^N (f_i^n m_i - f_i^n m_j) = 0$ we obtain $f_j^{N+n}(f_i^N m_i) - f_i^{N+n}(f_j^N m_j) = 0$, so we can write $f_j^N m_i - f_i^N m_j = 0$; we replace f_i by the powers, so we have simply $f_j m_i - f_i m_j = 0$.

Now use the fact that $\bigcup_i D(f_i) = X$ so $Z(f_1, \dots, f_r) = \emptyset$ and $\langle f_1, \dots, f_r \rangle = \langle 1 \rangle$, so $1 = \sum_i a_i f_i$, $a_i \in A$. Let $m = \sum_i a_i m_i$. Then

$$f_j m = \sum_i a_i f_j m_i = \sum_i a_i f_i m_j = m_j.$$

so $m = m_j / f_j \in M_{f_j}$, and α is surjective. $\square$

We have that $\widetilde{\ }$ is an exact functor (because localization is exact), and the association $M \mapsto \widetilde{M}$ and contrarily $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ when composed are the identity on A-modules, but when composed in the other order on sheaves is not, hence we do not quite have an equivalence of categories.

Example. If A is a ring, $X = \operatorname{Spec} A$, $U \subsetneq X$ any open subset. We take $\mathscr{F} = i_!(\mathscr{O}_X|_U) = i_!(\mathscr{O}_U)$, so $\Gamma(X, i_!\mathscr{O}_U) = 0$, but $\widetilde{0} \neq \mathscr{F}$.

We know (5.5), that every quasi-coherent $\mathscr{F}$ over X affine is obtained as $\mathscr{F} = \widetilde{M}$ for some A-module M. For example:

Example. Let $X = \mathbb{A}^1_k = \operatorname{Spec} A$, $A = k[x]$. If M is a finitely-generated A-module, $M \simeq A^r \oplus (\bigoplus_i A/(x-a_i)^{n_i})$ for $a_i \in k$. The sheaf $\widetilde{A/x^n}$ is the skyscraper sheaf at zero. If $K = K(X)$, $\widetilde{K}$ is a sheaf of $\mathscr{O}_X$-modules, where on $U = D(f)$ we have $K_f = K$ so $\widetilde{K} = \mathscr{K}$ is the constant sheaf. From

$$0 \to A \to K \to K/A \to 0$$

we obtain

$$0 \to \mathscr{O}_X \to \mathscr{K} \to \widetilde{K/A} \to 0.$$

Claim. $\widetilde{K/A} = \bigoplus_{P \in X} (i_P)_*(K/A_{\mathfrak{p}})$, the direct sum of skyscraper sheaves.

Note that $A_{\mathfrak{p}}$ is a DVR, consisting of $\{f \in K : v(f) \geq 0\}$; hence $K/A_{\mathfrak{p}} = \{f : v(f) < 0\}$.

We have seen that $\mathscr{F}$ is quasicoherent iff for all open affine sets U, $\mathscr{F}|_U \simeq \widetilde{M}$ for some M.

If $f : X \to Y$, and $\mathscr{F}$ is quasicoherent on Y, then $f^*\mathscr{F}$ is quasicoherent on X. If $f : X \to Y$, and either X is noetherian or f is separated or quasicompact, then f_* is quasicoherent. These hypotheses are necessary:

Example. Let $X = \bigsqcup_{i=1}^{\infty} \mathbb{A}^1_k$ be an infinite union of affine lines, $Y = \mathbb{A}^1_k$. We have $\Gamma(Y, f_*\mathscr{O}_X) = \prod_{i=1}^{\infty} A = \prod_{i=1}^{\infty} k[x]$, and thus if we take $U = \operatorname{Spec} A_f$, $\Gamma(U_f, f_*\mathscr{O}_X) = \prod_{i=1}^{\infty} A_f$. If $f_*\mathscr{O}_X$ were quasicoherent, we would have to have $\prod_{i=1}^{\infty} A = M$, but $M_f = (\prod_{i=1}^{\infty} A)_f \neq \prod_{i=1}^{\infty} A_f$, since a sequence $(a_1/f_1, a_2/f_2, \dots)$ cannot necessarily be written in the form $1/f(a_1, a_2, \dots)$.

As concerns $\mathscr{O}_X$-modules over a projective scheme, one can equivalently define $\widetilde{M}$ either by $\widetilde{M}(U)$ the set of sections to $\bigsqcup_{\mathfrak{p} \in U} (M_{\mathfrak{p}})_0$ that are locally fractions as is done in the book, or equivalently cover X by open affines $D_+(f)$, $f \in S$ homogeneous and $D_+(f) = \{\mathfrak{p} \not\ni f\} \simeq \operatorname{Spec}(S_f)_0$, and put on $D_+(f)$ the sheaf $\widetilde{(M_f)_0}$. The sheaves glue together as they must.

§II.6: Divisors and §II.7: Projective Morphisms

The theory of divisors is simplest when considered for curves. For example, here is a proof of (6.1) for curves:

Proposition (Lemma 6.1)**.** $v_P(f) = 0$ *for almost all* $P \in X$.

Proof. A rational function is defined as a regular function $f : U \to k = \mathbb{A}^1_k$ which if we consider $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ can be extended (I, 6.8) to a map $f : X \to \mathbb{P}^1_k$. Now $v_P(f) > 0$ iff $f \in \mathfrak{m}_P \subset \mathscr{O}_P$ iff $P \in f^{-1}(0)$, and $v_P(f) < 0$ iff $P \in f^{-1}(\infty)$. We need to show there are only finitely many in a preimage, which is a result of the following proposition. □

Proposition (Proposition 6.9)**.** *If $f : X \to Y$ is a dominant morphism of nonsingular curves, and $n = [K(X) : K(Y)]$, then for all $Q \in Y$, the divisor $f^*Q = \sum_{P \in f^{-1}(Q)} v_P(t_Q)P$ has degree n.*

Remark. This is the same theorem as in algebraic number theory, where if $A \subset K$ and L/K is a finite extension, B the integral closure of A in L, then $\sum_i e_i f_i = n = [L : K]$.

Proof. Let $V = \operatorname{Spec} B \subset Y$, and let A be the integral closure of B in $K(X)$. Localizing we have $\mathscr{O}_Q \subset K(Y)$ and its integral closure $\overline{\mathscr{O}_Q} \subset K(X)$, the latter of which is a torsion-free module over a PID and thus a free $\mathscr{O}_Q$-module of rank $[K(X) : K(Y)]$. Hence $\deg f^*Q = \dim_k A/\mathfrak{p}A$ (an Artin ring) where $B/\mathfrak{p} = k$ for some prime $\mathfrak{p} \subset B$. □

Example. If $X = \mathbb{P}^1_k$, $\operatorname{Div} X$ is a big group. If we identify $K(X) = k(x)$, $f = x - a$ for any $a \in k$ has a zero at a, and if $x = 1/y$, then $f = 1/y - a$ has a pole at ∞. Thus $\div f = (a) - (\infty)$, so $(a) \sim (\infty)$ for all $a \in k$. Thus $\operatorname{Pic} X \subset \mathbb{Z}(\infty)$. To see that $(\infty) \not\sim 0$, we note that $\deg(\infty) \neq 0$.

We define invertible sheaves on curves: if $D = \sum_P n_P P$ is effective ($n_P \geq 0$ for all P), we associate the closed subscheme $Z_D = \bigcup_P \mathscr{O}_{P,X}/t_P^{n_P}$ of dimension zero. In this way, there is a one-to-one correspondence between effective divisors and closed subschemes $Z_D \subset X$ of dimension 0, since if we have the stalk $I_P \subset \mathscr{O}_P$ of the given quasicoherent sheaf of ideals, then since $\mathscr{O}_P$ is a DVR, $I_P = \mathfrak{m}_P^n$, and thus we can associate the integer n to the divisor.

In fact, the associated quasicoherent sheaf to this closed subscheme is an invertible sheaf of $\mathscr{O}_X$-modules, since all stalks are contained in local rings which are PIDs, hence locally generated by $t_P^{n_P}$ at each P. So we define $\mathscr{L}(D) = \mathscr{I}_D^{-1} \subset \mathscr{K}$, locally generated by $t_P^{-n_P}$. A *sheaf of fractional ideals* is a locally finitely generated subsheaf of $\mathscr{O}_X$-modules of $\mathscr{K}$, and in this way we obtain a one-to-one correspondence between divisors D and fractional ideals $\mathscr{L}(D) \subset \mathscr{K}$ (given $\mathscr{L}$, for all P with local generator $s_P \in \mathscr{L}_P$, we associate $\sum_P v_P(s_P)P$, which is a finite sum because $\mathscr{L}$ is locally free and thus free on $U = \operatorname{Spec} A$, generated by $s \in K$ with only finitely many zeros and poles and there are only finitely many points missed). In this correspondence, two linearly equivalent divisors $D_1 \sim D_2$ correspond to isomorphic sheaves $\mathscr{L}_1 \simeq \mathscr{L}_2$, since if $D_1 - D_2 = \div f$, then $K \xrightarrow{f} K$ gives an isomorphism $\mathscr{L}_2 \to \mathscr{L}_1$ and conversely. So $\operatorname{Pic} X$ in the case of curves is also the group of invertible $\mathscr{O}_X$-modules modulo isomorphism.

Now we preview linear systems as an application of divisors on curves. Let X be a nonsingular projective plane curve, $V = S_1$ the k-vector space generated by x_0, x_1, x_2. For an element $l \in V$ we have a line $L_l \subset \mathbb{P}^2$ and vice versa. We can take $D_l = X \cap L_l$ as a divisor, with $\deg D_l = \deg X$. For any two such lines l_1, l_2, we can take the quotient of the $f = l_1/l_2$, so that $\div f = D_1 - D_2$, and thus $D_1 \sim D_2$.

We let $\mathfrak{d}$ be the family of linearly equivalent divisors D_l on X, indexed by the vector space $V = S_1$. $\mathfrak{d}$ is in one-to-one correspondence with $(V \setminus \{0\})/k^\times$, and thus is a "projective space". This is an example of a linear system as we will see.

From $(X, \mathfrak{d})$, we can recover the embedding $X \hookrightarrow \mathbb{P}^2$. As a set, map $X \to \mathbb{P}^2$ on a point $P \in X$ by considering the subvector space of lines such that $P \in D_l$; this is dimension 2 (it imposes one condition on a vector space of dimension 3), generated say by $y_{P,1}, y_{P,2}$, thus $I = \langle y_{P,1}, y_{P,2} \rangle \subset S$ is an ideal of the point $P \in \mathbb{P}^2$. This in fact a morphism: take x_0, x_1, x_2, and $U_i = \{x_i \neq 0\} \subset \mathbb{P}^2$, and $W_i = U_i \cap X$. We need to show that we have a morphism $W_0 \to U_0 = \operatorname{Spec} k[x_1/x_0, x_2/x_0]$, which will arise from a morphism $k[x_1/x_0, x_2/x_0] \to \Gamma(W_0, \mathscr{O}_{W_0})$ as follows.

If X is a curve, and D a divisor, then we have an invertible sheaf $\mathscr{L} = \mathscr{L}(D)$. If $s \in \Gamma(X, \mathscr{L})$, we get $s_p \in \mathscr{L}_P$, generated say by $f_P \in \mathscr{L}_P$, and if $s_P = g_P f_P$, for $g_p \in \mathscr{O}_P$, then $(\div s)_0 = \sum_P v_P(g_P)P$ and $(\div s)_0 \sim D$ because $s \in K$ has $(\div s)_0 - \div s = D$ (the difference at each point P is measured by g_P). In sum, to each global section $s \in \Gamma(X, \mathscr{L})$ we get an effective divisor $(\div s)_0 \sim D$, and conversely, if D' is an effective divisor $D' \sim D$, then D' arises in this way, since D' is effective iff $\mathscr{O}_X \subset \mathscr{L}(D') \simeq \mathscr{L}(D) \subset K$ is defined by the map $1 \mapsto s'$. Therefore the set $|D|$ of effective divisors $D' \sim D$ is in one-to-one correspondence with the set $(\Gamma(X, \mathscr{L}(D)) \setminus \{0\})/k^\times$.

Definition. We call $|D|$, the set of effective divisors $D' \sim D$ a *complete linear system.*

Therefore to $\mathscr{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$, $\mathscr{L} = \mathscr{O}_X(1)$ on X, so $V \subset \Gamma(X, \mathscr{O}_{\mathbb{P}^2}(1))$. So the morphism $k[x_1/x_0, x_2/x_0] \to \Gamma(W_0, \mathscr{O}_{W_0})$ on W_0, $x_0 \neq 0$, arises from the isomorphism $\mathscr{O}_X \simeq \mathscr{L}$ which globally is a map $1 \mapsto x_0$.

Definition. A *linear system* is a subset $\mathfrak{d} \subset |D|$ corresponding to a linear subspace of $(\Gamma(X, \mathscr{L}) \setminus \{0\})/k^\times$ (iff a subvector space of $\Gamma(X, \mathscr{L})$). We let $\dim \mathfrak{d}$ be its dimension as a projective space, $\dim V - 1$.

We would like to know, given X and $\mathfrak{d}$, can we find an embedding into a projective space $\mathbb{P}^n$?

Definition. A *base point* of the linear system $\mathfrak{d}$ is a point $P \in X$ such that $P \in \operatorname{Supp} D$ for all $D \in \mathfrak{d}$.

Base points do exist: we can for example take the linear system of lines through a point in $\mathbb{P}^2$.

It is a fact that $(X, \mathfrak{d})$ determine an embedding into $\mathbb{P}^n$ implies that $\mathfrak{d}$ has no base points. We can always just consider the open set U obtained by removing the set of base points of $\mathfrak{d}$ from X. We define a morphism $\phi : U \to \mathbb{P}(V) = \operatorname{Proj} S(V)$ where $S(V)$ is the symmetric algebra $S(V) = k \oplus V \oplus S^2(V) \oplus \dots$, where $S^2(V) = V \otimes V / \langle x \otimes y - y \otimes x \rangle_{x,y \in V}$. Warning: a point $P \in \mathbb{P}(V)$ corresponds to an ideal V' which is a codimension 1 subspace of the dual vector space.

Now it is worth surveying how to extend the theory of divisors to varieties other than nonsingular projective curves. For curves, we have equivalently:

(1) Divisors $D = \sum_P n_P P$, P closed points, modulo principal divisors $\div f = \sum_P v_P(f) P$.
(2) $Z_D \subset C$ closed subschemes associated to effective divisors (locally principal divisors, iff the associated quasicoherent sheaf $\mathscr{I}$ is locally generated by one element).
(3) Sheaves of fractional ideals $\mathscr{L} \subset \mathscr{K}$.
(4) Invertible sheaves $\mathscr{L}$.

As a comment for (Ex. 6.5.2), we can also note that the prime ideal $\langle x, z \rangle$ cannot be principal because $k[x,y,z]/\langle xy - z^2 \rangle$ is not a UFD ($xy = zz$, all of degree 1 hence irreducible).

For more general varieties, there are several different generalizations of divisors and linear equivalence. For (1), we may generalize to Weil divisors if X is normal (integral) variety over a field $k = \overline{k}$ (iff $\mathscr{O}_P$ is normal for all P iff X is regular in codimension 1). In this generality, Weil divisors will not always correspond to invertible sheaves. For (2), we have the notion of *Cartier divisors*. The notions in (4) generalize in an obvious way.

For (3), we also have *generalized divisors*. If X is a scheme which is G_1 (Gorenstein in codimension 1), and S_2 (Serre's condition). (For example, for curves, normal is equivalent to nonsingular, but any plane curve is G_1. For surfaces, nonsingular implies normal implies isolated singularities, but any surface in $\mathbb{P}^3$ is G_1.) We consider $\mathscr{O}_X \subset \mathscr{K}_X$ and $\mathscr{L} \subset \mathscr{K}_X$ a fractional ideal. We require $\mathscr{L}$ to be *reflexive*, i.e. $\mathscr{L} \simeq (\mathscr{L}^\vee)^\vee$, where $\mathscr{L}^\vee = \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$, plus some other technical conditions. Here we have $\operatorname{CaCl} X \simeq \operatorname{Pic} X$ under a weak hypothesis, and

$$\operatorname{CaCl} X \subset \operatorname{APic} X \subset \operatorname{GPic} X$$

where $\operatorname{APic} X$ is the group of almost Cartier divisors (Cartier except on subsets of codimension ≥ 2) since $\operatorname{GPic} X$, the set of reflexive fractional ideals is not quite a group. For more information on this, see Robin Hartshorne, *Generalized divisors on Gorenstein schemes*, K-Theory **8** (1994), no. 3, 287–339.

As an example, we have the following:

Proposition. *If X is G_1 and S_2, then $APic\, X = GPic\, X$ if and only if X is normal.*

Example. Let $X = H_1 \cup H_2$. We have

$$0 \to \operatorname{APic} X \to \operatorname{Pic} H_1 \oplus \operatorname{Pic} H_2 \oplus \operatorname{Div} L \to \operatorname{Pic} L \to 0.$$

The first arrow sends $D \mapsto (C_1, C_2, C_1 \cap L - C_2 \cap L)$ where $C_1 = D \cap H_1$, $C_2 = D \cap H_2$, and L is a linear section. We have $\operatorname{Pic} H_1 = \operatorname{Pic} H_2 = \mathbb{Z}$ and $\operatorname{Pic} L = \mathbb{Z}$. The second arrow sends the triple $(C_1, C_2, D) \mapsto C_1 \cap L - C_2 \cap L - D$. $\operatorname{GPic} X$ is a set together with the action of the group $\operatorname{APic} X$. The orbits are D, L, so it is a disjoint union.