TORIC SURFACES AND CONTINUED FRACTIONS

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One goal in studying toric varieties is to put general theory in more concrete, combinatorial terms. It is striking that when one considers toric surfaces (obtained from fans in the plane) that continued fractions—an object originating in number theory—pop up in the resolution of singularities. The purpose of this article is to provide an exposition of this phenomenon and other ways in which continued fractions arise in toric geometry.

1. Quotient singularities

The material in this section follows Fulton [6, §2.2, pp. 31–35].

1.1. **Cyclic quotients.** Consider the cyclic group $G = \mu_m \simeq \mathbb{Z}/m\mathbb{Z}$ of mth roots of unity acting on $\mathbb{C}[U,V]$ via $U \mapsto \zeta U$, $V \mapsto \zeta V$ where $\zeta^m = 1$ is primitive. We would like to find the ring of invariants $\mathbb{C}[U,V]^G$. Since G acts on monomials as $U^i V^j \mapsto \zeta^{i+j} U^i V^j$, after comparing coefficients we conclude that the invariants are generated by monomials $U^i V^j$ such that $m \mid (i+j)$, therefore

$$\mathbb{C}[U,V]^G = \mathbb{C}[U^m, U^{m-1}V, \dots, UV^{m-1}, V^m] = \mathbb{C}[U,V]_m \hookrightarrow \mathbb{C}[U,V].$$

If we let $X = U^m$, $Y = V/U$, this ring becomes $\mathbb{C}[X, XY, \dots, XY^m]$, the cone over the *rational (normal) curve of degree m*. This is indeed a toric variety, obtained via the cone σ generated by $v_1 = me_1 - e_2 = (m,-1)$ and $v_2 = e_2 = (0,1)$ in the plane $N_{\mathbb{R}} = \mathbb{R}^2$, $N = \mathbb{Z}^2$, as in Figure 1.

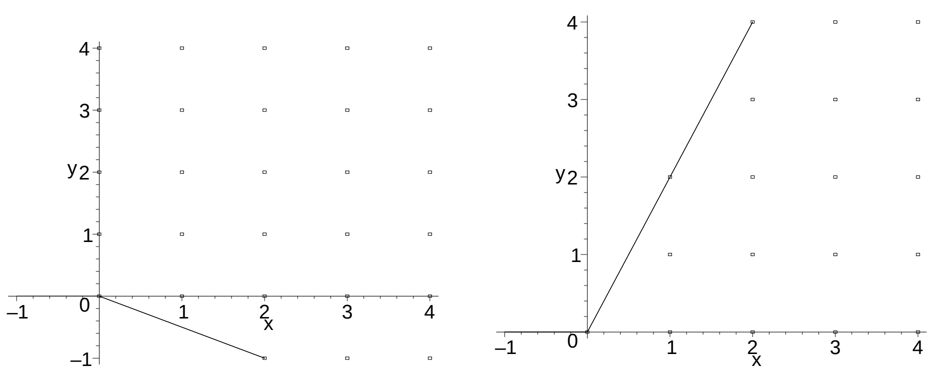

Figure 1. $\sigma \subset N$ and $\sigma^\vee \subset M$, $m = 2$

For $m = 2$, we obtain the familiar cone over the rational cubic curve (a quadric surface),

$$A_\sigma = \mathbb{C}[X, XY, XY^2] = \mathbb{C}[A,B,C]/\langle B^2 - AC \rangle.$$

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Equivalently, we can describe the variety $U_\sigma = \operatorname{Spec}(A_\sigma)$ by letting G act on $\mathbb{C}^2$ by $(u, v) \mapsto (\zeta u, \zeta v)$ to obtain the *cyclic quotient singularity* $\mathbb{C}^2/G$. This equivalence follows from the fact that the invariants (the elements of $\mathbb{C}[U, V]^G$) are exactly those polynomial functions which are constant on the orbits of G and therefore they form the ring of functions on the quotient $\mathbb{C}^2/G$. (For more on algebraic groups and quotient actions, see [19, Chapter III, §4].)

There is a natural toric interpretation of this construction. Let N' be the lattice generated by the rays v_1, v_2 of σ, and let σ' be the same cone as σ considered in N'. Recall ([6, §1.3, p. 18]) that if a homomorphism of lattices $N' \to N$ maps a cone $\sigma' \subset N'$ to a cone $\sigma \subset N$, then the dual $M \to M'$ induces a map of semigroups $S_\sigma \to S_{\sigma'}$ and a ring homomorphism $A_\sigma \to A_{\sigma'}$. In our case, since N' has a basis me_1, e_2, the dual lattice $M \subset M'$ is generated by $(1/m)e_1^*$, e_2^* which corresponds to monomials U, Y such that $U^m = X$:

$$\mathbb{C}[M] = \mathbb{C}[U, Y, (UY)^{-1}], \quad \mathbb{C}[M'] = \mathbb{C}[X, Y, (XY)^{-1}].$$

The generators for $S_{\sigma'}$ are $(1/m)e_1^*$ and $(1/m)e_1^* + e_2^*$—indeed for $m = 2$, we have as in Figure 2.

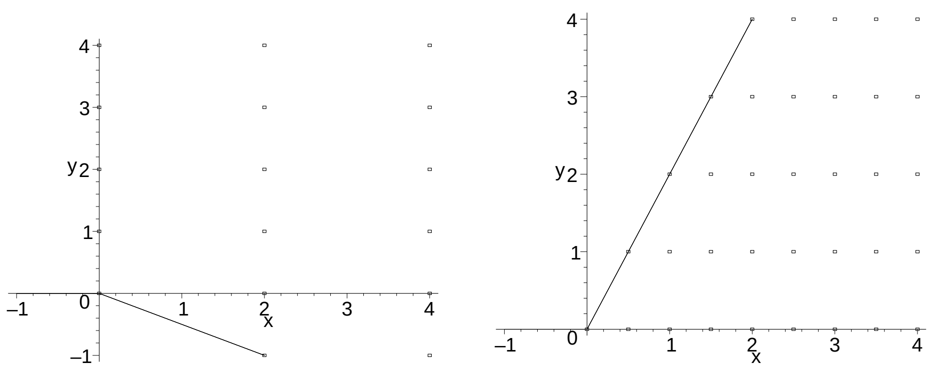


FIGURE 2. $\sigma' \subset N'$ and $(\sigma')^\vee \subset M'$

Therefore $A_{\sigma'} = \mathbb{C}[U, UY]$, and it is no surprise that $U_{\sigma'} \simeq \mathbb{C}^2$ because it is generated by a basis for the lattice N'. Indeed, we have the induced lattice homomorphism $M \subset M'$ which begets a semigroup homomorphism and corresponding ring map

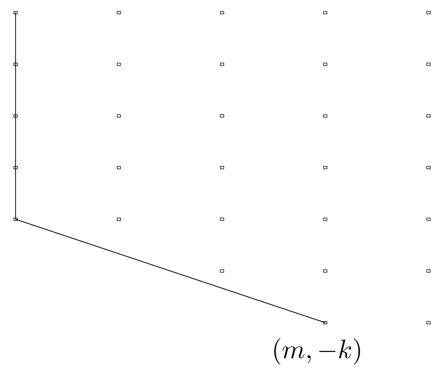
$$A_\sigma = \mathbb{C}[X, XY, \dots, XY^m] \hookrightarrow \mathbb{C}[U, UY] = A_{\sigma'}$$

where again $U^m = X$. The substitution $Y = V/U$ identifies this injection with the map $\mathbb{C}[U^m, \dots, V^m] \hookrightarrow \mathbb{C}[U, V]$ above.

What we have just shown is that the inclusion of invariants in the polynomial ring corresponds to a contraction of the lattice for a certain prescribed cone, or to state it another way, the toric surface is in fact a quotient variety.

1.2. **Generalization to Two-Dimensional Toric Varieties.** A similar identification holds for any (singular) two-dimensional affine toric variety. In order to show this, we first prove:

Lemma 1.1. *Any two-dimensional affine toric variety comes from a cone σ generated by $v_1 = e_2 = (0,1)$ and $v_2 = me_1 - ke_2 = (m,-k)$ with $0 \leq k < m$ and* $\gcd(m,k) = 1$*:*



Proof. Since by assumption σ is generated by two rays in the plane, any minimal generator along one of these rays will be part of a basis for N so we may assume that (under an appropriate linear change of coordinates) it is in fact $v_1 = (0,1)$. After a reflection in the y-axis (if necessary) we may take the second to be $v_2 = (m,y)$ for $m \geq 0$. Furthermore, we can apply the automorphism $(x,y) \mapsto (x, px+y)$ for an integer p; this will fix v_1 and allow us to change y by multiples of m to bring it in the range $y = -k$, $0 \leq k < m$. The statement that $\gcd(m,k) = 1$ follows by taking a minimal generator along v_2.

Specifically, given the primitive generators $(a,b), (c,d)$, we find integers w, z such that $aw + bz = 1$ by the Euclidean algorithm, and apply the automorphism

$$\begin{pmatrix} -b & a \\ w & z \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & ad-bc \\ 1 & wc+zd \end{pmatrix}$$

to obtain $v_1 = (0,1)$. Taking the x-coordinate of the second generator to be > 0 by reflection equates to taking its absolute value. With $m = |ad-bc|$, $y = wc+zd$, we then find the integer p such that $0 \leq -(pm+y) < m$ (this is in fact $p = -\lfloor y/m \rfloor - 1$ whenever $m \neq 1$, $p = -y$ when $m = 1$) and apply the automorpism

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 0 & m \\ 1 & y \end{pmatrix} = \begin{pmatrix} 0 & m \\ 1 & pm+y \end{pmatrix}.$$

The composition of these two gives directly the desired lattice automorphism. □

This can be easily be implemented in Maple as follows:

Algorithm 1.2 (Computing standard form).

```
with(linalg):
# Takes as argument two generators for a
# cone as columns of a matrix; prints an
# automorphism of the lattice and the
# generators in standard form
stdform := proc (A) local i,j,g,w,z,M,p,E,T;
  # Ensure generators are primitive
  gcd( A[1,1],A[2,1], 'A[1,1]','A[2,1]' );
  gcd( A[1,2],A[2,2], 'A[1,2]','A[2,2]' );
```

```
  # Euclidean algorithm
  igcdex( A[1,1],A[2,1], 'w','z' );
  E := array([[-A[2,1],A[1,1]],[w,z]]);
  A := multiply(E,A);
  # Reflection
  if ( A[1,2]<0 ) then
    A[1,2] := abs(A[1,2]);
    E := multiply(array([[-1,0],[0,1]]),E);
  fi;
  # Translation
  p := -floor(A[2,2]/A[1,2])-1;
  if ( p <> 1 ) then
    p := p-1;
  fi;
  T := array([[1,0],[p,1]]);

  # Output
  multiply(T,E), multiply(T,A);
end:
```

As an example, the command `stdform( array([[3,6],[5,4]]) )` gives as output the automorphism $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$ and standard generators $(0,1),(9,-5)$.

With σ in this standard form, we let N' be generated by $v_1 = me_1 - ke_2, v_2 = e_2$ (or just me_1, e_2). M' is again generated by $(1/m)e_1^*, e_2^*$ corresponding to monomials U, Y, so $S_{\sigma'}$ is generated by $(1/m)e_1^*$ and now $k(1/m)e_1^* + e_2^*$. Therefore $A_{\sigma'}$ is $\mathbb{C}[U, U^kY] = \mathbb{C}[U,V]$ with $Y = V/U^k$, and the map $A_\sigma \hookrightarrow A_{\sigma'}$ is induced by the inclusion of lattices.

To see how this arises from invariants, we notice that we can modify the group action to $G = G_{m,k} = \mu_m$ acting on $\mathbb{C}^2$ by $(u,v) \mapsto (\zeta u, \zeta^k v)$; the corresponding variety $\mathbb{C}^2/G$ has the ring of functions $\mathbb{C}[U,V]^G$. We will prove the following claim:

Claim. For σ in this standard form, $A_\sigma = \mathbb{C}[U,V]^{G_{m,k}}$ so $U_\sigma = \mathbb{C}^2/G_{m,k}$.

1.3. **Computing Invariants.** Thankfully, the problem of computing invariants of a polynomial ring under the action of subgroups of $GL_n(\mathbb{C})$ is a classical problem for which there are nice treatments. An introduction to this subject is given by [4, §7.3]; the general situation is handled in [21].

In our case, we have the finite subgroup $G = G_{m,k}$, which acts as $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^k \end{pmatrix}$ on $\mathbb{C}^2$. It follows from a theorem of Noether that the ring of invariants will be generated by elements of degree $\leq \#G = m$. A quick computation shows that U^m, V^m are fixed by G. One can obtain a full generating set by adding the elements $R_G(U^iV^j)$, $0 \leq i < m$, $0 \leq j < m$, where R_G is the *Reynolds operator*

$$R_G(\phi(U,V)) = \frac{1}{\#G} \sum_{g \in G} {}^g\phi(U,V).$$

Following [13] (also see the English translation [14]), we find

$$R_G(U^iV^j) = \frac{1}{m}\left(\sum_{\nu=0}^{m-1} \zeta^{\nu(i+kj)}\right) U^iV^j = \frac{1}{m}\left(\frac{1-(\zeta^{i+kj})^m}{1-\zeta^{i+kj}}\right)U^iV^j.$$

Since $\zeta^m = 1$, the sum will vanish whenever the denominator is nonzero, i.e.

$$R_G(U^iV^j) = \begin{cases} (1/m)(m)U^iV^j = U^iV^j, & i + qj \equiv 0 \pmod{m}, \\ 0, & \text{otherwise.} \end{cases}$$

We have proven:

Lemma 1.3. *$\mathbb{C}[U, V]^G$ is generated by those monomials U^iV^j with $i + kj \equiv 0 \pmod{m}$, $0 \le i \le m$, $0 \le j \le m$, $(i, j) \ne (m, m)$.*

The following argument seems also to suffice: since the action of G only affects coefficients, the invariants will be generated by monomials as we saw above, and since $U^iV^j \mapsto \zeta^{i+kj}U^iV^j$, the result follows.

This computation does not guarantee that the generating set is minimal. Indeed, if $k = m - 1$ then $i \equiv j \pmod{m}$ so one has the invariants

$$1, UV, U^2V^2, \ldots, U^{m-1}V^{m-1}, U^m, V^m$$

for which $1, UV, U^m, V^m$ clearly suffice. This problem will be taken up in the next section—it is closely related to resolving the singularities of the corresponding toric variety.

This is however enough to conclude our discussion of invariants. For a cone σ in standard form (generated by $(0, 1), (m, -k)$) we have that $\sigma^\vee$ is generated by $(1, 0), (k, m)$ so that S_σ is generated by these elements and points (i, j) such that $j/i \le m/k$ (taking slopes). Therefore A_σ is generated by the monomials X, X^kY^m, and X^iY^j where $kj \le mi$. If we make the substitution $X = U^m$, $Y = V/U^k$, we find generators U^m, V^m and $U^{mi-kj}V^j$. Since $mi - kj + k(j) \equiv 0 \pmod{m}$, and $mi - kj \ge 0$, this monomial is invariant under G. Contrarywise, a monomial U^iV^j with $i + kj \equiv 0 \pmod{m}$ can be written $U^{i'm-kj}V^j = X^{i'}Y^j$ with $i'm - kj \ge 0$. Therefore indeed $A_\sigma = \mathbb{C}[U, V]^G$, and $U_\sigma = \mathbb{C}^2/G$ as was claimed in the preceding section.

There is another way to see this. The action of G naturally extends to the entire Laurent ring

$$\mathbb{C}[U, V, (UV)^{-1}] = \mathbb{C}[U, Y, (UY)^{-1}].$$

Notice that $Y = V/U^k$ is fixed by G in this map, so that the invariants are

$$\mathbb{C}[U, U^{-1}, Y, Y^{-1}]^G = (\mathbb{C}[U, U^{-1}])^G[Y, Y^{-1}] = \mathbb{C}[X, Y, (XY)^{-1}],$$

since $X = U^m$. From this, we find

$$\begin{aligned} A_\sigma &= A_{\sigma'} \cap \mathbb{C}[M] = A_{\sigma'} \cap \mathbb{C}[X, Y, (XY)^{-1}] \\ &= A_{\sigma'} \cap \mathbb{C}[U, V, (UV)^{-1}]^G = A_{\sigma'} \cap \mathbb{C}[M']^G = (A_{\sigma'})^G = \mathbb{C}[U, V]^G, \end{aligned} \tag{1}$$

which is just what we proved above.

To summarize, let us state the result so far as follows:

Proposition 1.4. *Any two-dimensional toric variety has only quotient singularities (such a variety is referred to as an* orbifold *or V*-manifold*). For each maximal cone σ, the corresponding affine open U_σ can be written as a quotient of $\mathbb{C}^2$ by the action of a finite cyclic group $G_{m,k}$ and the inclusion $\mathbb{C}^2 \hookrightarrow U_\sigma$ is induced by the inclusion of the lattice generated by the rays of σ into $N = \mathbb{Z}^2$.*

Remark. Brieskorn [3] categorized all subgroups of $GL_2(\mathbb{C})$ which give quotient singularities: together with the cyclic group above, the dihedral group and symmetries of the tetrahedron, octahedron, and icosahedron occur. Riemenschneider [17] calculated the minimal invariants for each of these subgroups.

1.4. **Generalization to Higher Dimension.** This result can be extended under certain circumstances to higher dimensional toric varieties. For a lattice N of any rank, let $N' \subset N$ be a sublattice of finite index with $M \subset M'$ the dual lattices. We have a canonical pairing

$$M'/M \times N/N' \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$$

by the composition of the dual pairing (scaled to $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$) together with the map $q \mapsto \exp(2\pi i q)$. This defines an action of $G = N/N'$ (a finite abelian group) on $\mathbb{C}[M']$ by

$$v(X^{u'}) = \exp(2\pi i \langle u', v\rangle) X^{u'}$$

for $v \in N$, $u' \in M'$. Via the above action of G on $\mathbb{C}[M']$, we have:

Claim. $\mathbb{C}[M']^G = \mathbb{C}[M]$.

Proof. Since N/N' is finite we may find a basis $e_1, \dots, e_n$ for N so that N' is generated by $m_1 e_1, \dots, m_n e_n$ for integers $m_i > 0$. Then

$$\mathbb{C}[M'] = \mathbb{C}[U_1, \dots, U_n, (U_1 \dots U_n)^{-1}], \quad \mathbb{C}[M] = \mathbb{C}[X_1, \dots, X_n, (X_1 \dots X_n)^{-1}]$$

with $U_i^{m_i} = X_i$.

An element $(a_1, \dots, a_n) \in N/N' = \bigoplus_i \mathbb{Z}/m_i\mathbb{Z}$ acts on monomials by multiplying $U^\ell = U_1^{\ell_1} \dots U_n^{\ell_n}$ by the scalar $\exp(2\pi i (\sum_i a_i \ell_i / m_i))$. Since we may take the element $(0, \dots, 0, 1, 0, \dots, 0) \in N/N'$ for index i, we must have $\ell_i/m_i \in \mathbb{Z}$ for each i (i.e. $m_i \mid \ell_i$), and $\mathbb{C}[M']^G = \mathbb{C}[M]$ as claimed. □

In the special case when N has rank 2 and N' is generated by me_1 and e_2, N/N' is isomorphic to μ_m and we have the action as described above.

If σ is an n-simplex in $N_{\mathbb{R}}$ (i.e. σ is generated by n independent vectors), we let $N' \subset N$ be the sublattice generated by the primitive elements in $\sigma \cap N$. We have a cone $\sigma' \subset N'$ with a map $\mathbb{C}^n = U_{\sigma'} \to U_\sigma$; the abelian group $G = N/N'$ acts on $U_{\sigma'}$ with

$$U_\sigma = U_{\sigma'}/G = \mathbb{C}^n/G$$

after intersecting $A_{\sigma'}$ with $\mathbb{C}[M']^G = \mathbb{C}[M]$ as in (1).

The key property we need is that σ is *simplicial*, i.e. σ is generated by linearly independent vectors.

Proposition 1.5. *A toric variety given by a simplicial fan has only quotient singularities. For each maximal cone σ, the affine open U_σ is the quotient of $\mathbb{C}^n$ by the action of the finite abelian group $G = N/N'$ where N' is the lattice obtained by the primitive generators of σ.*

In general, a cone that is not maximal is the product of a torus and a quotient of $\mathbb{C}^d$ for some $d \leq n$ (by expanding the primitive generators to all of $\mathbb{C}^n$).

As illustrations of this proposition, we complete the two exercises [6, p. 35].

Proposition 1.6. *Let σ be the cone generated by*

$$e_1, e_2, \ldots, e_{n-1}, -e_1 - e_2 - \cdots - e_{n-1} + me_n$$

where $e_1, \ldots, e_n$ are a basis for N. Then:

(a) $U_\sigma = \mathbb{C}^n/\mu_m$ *where the mth roots of unity μ_m act by*

$$(u_1, \ldots, u_n) \mapsto (\zeta u_1, \ldots, \zeta u_n);$$

(b) U_σ *is the cone over the m-tuple Veronese embedding of $\mathbb{P}^{n-1}_{\mathbb{C}}$.*

Proof. Since σ is simplicial, we may take the lattice N' and cone σ' obtained from the generators of σ. Since N' is also generated by $e_1, \ldots, e_{n-1}, me_n$, we find that $N/N' \simeq \mathbb{Z}/m\mathbb{Z} \simeq \mu_m$. Therefore M' has as a basis $e_1^*, \ldots, e_{n-1}^*, (1/m)e_n^*$ corresponding to monomials $U_1 = X_1, \ldots, U_{n-1} = X_{n-1}$ and $U_n^m = X_n$, where

$$\mathbb{C}[M'] = \mathbb{C}[U_1, \ldots, U_n, (U_1 \ldots U_n)^{-1}], \quad \mathbb{C}[M] = \mathbb{C}[X_1, \ldots, X_n, (X_1 \ldots X_n)^{-1}].$$

The generators for S'_σ arise from

$$e_1^* + (1/m)e_n^*, \ldots, e_{n-1}^* + (1/m)e_n^* \text{ and } (1/m)e_n^*$$

(calculated using the "practical procedure" [6, p. 11]). Thus

$$A_{\sigma'} = \mathbb{C}[U_1U_n, \ldots, U_{n-1}U_n, U_n]$$

which via the substitution $U_1 = V_1/U_n, \ldots, U_{n-1} = V_{n-1}/U_n$ and $V_n = U_n$ becomes

$$A_{\sigma'} = \mathbb{C}[V_1, \ldots, V_{n-1}, V_n]$$

so $U_{\sigma'} = \mathbb{C}^n$ as expected. We already have shown in the preceding discussion (Proposition 1.5) that $U_\sigma = U_{\sigma'}/G = \mathbb{C}^2/G$; the action is generated by

$$U_1 \mapsto U_1, \ldots, U_{n-1} \mapsto U_{n-1}, U_n \mapsto \zeta U_n$$

which under our substitution ($V_i = U_iU_n$) becomes

$$V_1 \mapsto \zeta V_1, \ldots, V_n \mapsto \zeta V_n$$

so we have the desired map on the coordinate ring, switching v for u. This proves (a).

The mth Veronese embedding of $\mathbb{P}^{n-1}$ is obtained as follows: we let $u_0, \ldots, u_n$ be coordinates on $\mathbb{P}^{n-1}$ and take the $N = \binom{n+m}{m}$ coordinates $v_{i_0 \ldots i_n}$ such that $i_0 + \cdots + i_n = m$ for nonnegative indices, with $v_{i_0 \ldots i_n} = u_0^{i_0} \ldots u_n^{i_n}$. In other words, we coordinatize the monomials in n variables of degree m. For example, with $m = 2$ and $n = 3$ we have

$$(u_0 : u_1 : u_2) \mapsto (v_{200} : v_{110} : \cdots : v_{002}) = (u_0^2 : u_0u_1 : u_0u_2 : u_1^2 : u_1u_2 : u_2^2).$$

We therefore obtain an embedding into $\mathbb{C}^N$ by the canonical embedding of projective space in the large affine space (taking cones); the image of the map is the variety

$$\operatorname{Spec}(\mathbb{C}[V_{i_0 \ldots i_n}]_{i_0 + \cdots + i_n = m})/\langle V_{i_0 \ldots i_n} - U_0^{i_0} \ldots U_n^{i_n} \rangle = \operatorname{Spec}(\mathbb{C}[U_0, \ldots, U_n]_m).$$

Since G operates on $\mathbb{C}[U_0, \ldots, U_n]$ by multiplying a monomial $U_0^{i_0} \ldots U_n^{i_n}$ by $\zeta^{i_0 + \cdots + i_n}$, we have as above that

$$\mathbb{C}[U_0, \ldots, U_n]^G = \mathbb{C}[U_0, \ldots, U_n]_m,$$

generated by the elements of degree m. We therefore have $A_\sigma = (A_{\sigma'})^G = \mathbb{C}[U_0, \ldots, U_n]^G = \mathbb{C}[U_0, \ldots, U_n]_m$ as claimed.

We remark that the mth Veronese image of the projective line $\mathbb{P}^1$ in $\mathbb{P}^m$ is the rational normal curve of degree m, so this generalizes the results obtained in §1.1. $\square$

Proposition 1.7. *If m and $a_1, \dots, a_n$ are positive integers, the quotient of $\mathbb{C}^n$ by the cyclic group μ_m acting by*

$$(u_1, \dots, u_n) \mapsto (\zeta^{a_1} u_1, \dots, \zeta^{a_n} u_n),$$

can be constructed as an affine toric variety U_σ by taking

$$N' = \sum_{i=1}^{n} \mathbb{Z}(1/a_i)e_i \subset N = N' + \mathbb{Z}(1/m)(e_1 + \dots + e_n),$$

and the cone σ generated by $e_1, \dots, e_n$. When $a_1 = \dots = a_n = 1$, this agrees with the construction of the preceding proposition.

Proof. By construction, $N/N' \simeq \mu_m$, generated by $w = (1/m)(e_1 + \dots + e_n)$. The dual lattice M' has generators $a_i e_i^*$ which agree with the generators for $S_{\sigma'}$, so

$$A_{\sigma'} = \mathbb{C}[U_1^{a_1}, \dots, U_n^{a_n}] = \mathbb{C}[V_1, \dots, V_n].$$

By the above, the group action is generated by the element w, which acts by multiplication on a monomial $U_1^{j_1} \dots U_n^{j_n}$ by $e^{2\pi i (j_1 + \dots + j_n)/m} = \zeta^{j_1 + \dots + j_n}$. It therefore acts on $A_{\sigma'} = \mathbb{C}[V_1, \dots, V_n]$ by

$$V_i = U_i^{a_i} \mapsto \zeta^{a_i} V_i,$$

and by the general setup above, $U_\sigma = U_{\sigma'}/G = \mathbb{C}^n/G$ as claimed.

In the case that $a_1 = \dots = a_n = 1$, we can take the basis $f_1, \dots, f_n$ for N where $f_i = e_i$ for $i \neq n$, $f_n = (1/m)(e_1 + \dots + e_{n-1} + e_n)$. Then N' has a basis $f_1, \dots, mf_n$, and σ is generated by $f_1, \dots, f_{n-1}$ and $-f_1 - \dots - f_{n-1} + mf_n$ as in the previous exercise. $\square$

This can sometimes but not always be extended to nonaffine toric varieties—we refer the reader to [6, pp. 35–36] and [11, pp. 35–37] for the details.

2. Minimal resolution of singularities on toric surfaces

We left the task of finding a minimal set of generators for the invariants incomplete from the previous section. We will do this first in a purely algebraic way and then reprove this in a toric setting, relating it to resolution of singularities. For the remainder of this section, we will fix the group $G = G_{m,k}$ acting as above.

2.1. Finding minimal generators algebraically.

The material in this section follows Riemenschneider [13, 14]. We begin with an example.

Example 2.1. Let us take $m = 5$, $k = 3$ so that the group action is $U, V \mapsto \zeta U, \zeta^3 V$ with $\zeta^5 = 1$. Lemma 1.3 says that

$$\mathbb{C}[U, V]^G = \mathbb{C}[U^5, U^2V, U^4V^2, UV^3, U^3V^4, V^5];$$

to compute U^iV^j, we let j range over $0, \dots, 5$ and find the unique $0 \le i \le m$ such that $i + 3j \equiv 0 \pmod 5$. But it is clear that $U^4V^2 = (U^2V)^2$ and $U^3V^4 = (U^2V)(UV^3)$ are superfluous, and that

$$\mathbb{C}[U, V]^G = \mathbb{C}[U^5, U^2V, UV^3, V^5]$$

is a minimal generating set.

This example is very illustrative: notice that as ordered, the minimal generators correspond to set of decreasing exponents on U. Indeed, if $U^{s_i}V^{t_i}$, $U^{s_j}V^{t_j}$ occur with $s_i \geq s_j$, $t_i < t_j$ then by an inductive argument on the exponent of V we can write U^iV^j as a product of generators and hence it is superfluous.

We can make this argument precise as follows: since the first two (minimal) generators are $U^m, U^{m-k}V$, we define the integers

$$s_1 = m, \quad s_2 = m-k, \quad t_1 = 0, \quad t_2 = 1$$

and we would like to know the next integer $t_3 > t_2$ such that the unique solution to $s_3 + kt_3 \equiv 0 \pmod{m}$ has $s_3 < s_2$. To do this, we find a natural number $b_2 \geq 2$ such that

$$s_1 = b_2 s_2 - s_3, \quad 0 \leq s_3 < s_2;$$

if we use the ordinary division algorithm, we can find an integer such that $s_1 = b_2' s_2 + s_3'$ where $0 < s_3' \leq s_2$ (notice this change in how we take the remainder) and $b_2' \geq 1$ (since $s_1 > s_2$); we then take $b_2 = b_2' + 1$ and $s_3 = s_3' - s_2$. If we continue this process, we obtain a sequence of numbers

$$\begin{aligned} s_1 &= b_2 s_2 - s_3 \\ s_2 &= b_3 s_3 - s_4 \\ &\vdots \\ s_{e-2} &= b_{e-1} s_{e-1} - s_e \end{aligned}$$

with the property that $b_2, \ldots, b_{e-1} \geq 2$ and

$$s_1 = m > s_2 = m-k > \cdots > s_{e-2} > s_{e-1} = 1 > s_e = 0. \tag{2}$$

We find that

$$\begin{aligned} \frac{m}{m-k} = \frac{s_1}{s_2} = b_2 - \frac{s_3}{s_2} = b_2 - \cfrac{1}{\cfrac{s_2}{s_3}} = b_2 - \cfrac{1}{b_3 - \cfrac{s_4}{s_3}} \\ = \cdots = b_2 - \cfrac{1}{b_3 - \cfrac{1}{\cdots - \cfrac{1}{b_{e-1}}}} = [[b_2; b_3, \ldots, b_{e-1}]]. \end{aligned}$$

Thus we have constructed a special continued fraction, known as a *Hirzebruch-Jung continued fraction*. Jung [8] considered curves on analytic surfaces and sought to characterize the functions defined in a neighborhood of singular points. Hirzebruch [7] then extended his results, considering abstract Riemann surfaces which failed to be well-coordinatized at a point and introduced the continued fraction when considering quotient varieties.

Remark. The Hirzebruch-Jung construction differs from the usual continued fraction for which all of the minus signs are replaced by the plus signs. The latter regular continued fractions arise in number theory in, for example, the calculation of the fundamental unit of a real quadratic field and thus solutions to Pell's equation, approximations to the values of roots of algebraic equations, etc.

Conversely, if we are given integers $b_2, \dots, b_{e-1} \geq 2$, we can inductively calculate the s_i via

$$s_1 = m, \quad s_2 = m - k, \quad s_{i+1} = b_i s_i - s_{i-1}, \quad i = 2, \dots, e-1.$$

Thus there is a one-to-one correspondence between these types of finite continued fractions and their rational equivalents.

Therefore we define $t_{i+1} = b_i t_i - t_{i-1}$ for the b_i computed above, and we also define

$$u_1 = 1, \quad u_2 = 1, \quad u_{i+1} = b_i u_i - u_{i-1}.$$

Example 2.2. Continuing the above ($m = 5$, $k = 3$), we have

$$\begin{gathered}
s_1 = 5, \quad s_2 = 2, \\
b_2 = \left\lfloor \frac{5}{5-3} \right\rfloor + 1 = 3, \quad s_3 = 3(2) - 5 = 1 \\
b_3 = \left\lfloor \frac{2}{1} \right\rfloor = 2, \quad s_4 = 2(1) - 2 = 0 \\
t_1 = 0, \quad t_2 = 1, \quad t_3 = 3(1) - 0 = 3, \quad t_4 = 2(3) - 1 = 5 \\
u_1 = 1, \quad u_2 = 1, \quad u_3 = 3(1) - 1 = 2, \quad u_4 = 2(2) - 1 = 3
\end{gathered}$$

Notice that this provides exactly the minimal generators.

Motivated by the above, we investigate the relationships between these integers.

Lemma 2.3. *We have:*

$$\begin{gathered} u_1 \leq u_2 \leq \cdots \leq u_e \\ t_1 < t_2 < \cdots < t_e \end{gathered} \tag{3}$$

$$m u_i = s_i + k t_i, \quad i = 1, \dots, e \tag{4}$$

$$\left.\begin{aligned} s_i t_{i+1} - s_{i+1} t_i &= m \\ s_i u_{i+1} - s_{i+1} u_i &= k \\ t_{i+1} u_i - t_i u_{i+1} &= 1 \end{aligned}\right., \quad i = 1, \dots, e-1. \tag{5}$$

Proof. The first set of equations follows by induction since

$$t_{i+1} = b_i t_i - t_{i-1} > 2t_i - t_{i-1} = t_i + (t_i - t_{i-1}) > t_i,$$

with a similar statement for u_i. The second equation follows from $mu_1 = m = m + 0 = s_1 + t_1$, $mu_2 = m = (m-k) + k = s_2 + kt_2$ and

$$m u_{i+1} = m(b_i u_i - u_{i-1}) = b_i(s_i + kt_i) - (s_{i-1} + kt_{i-1}) = s_{i+1} + kt_{i+1}.$$

The others follow similarly. □

The last equation of (5) implies that $\gcd(t_i, u_i) = 1$. The target is of course the decreasing values of t_i expressed in (3).

The example above suggests the following theorem:

Theorem 2.4. $\mathbb{C}[U,V]^G$ *is generated by the elements* $U^{s_i} V^{t_i}$ *for* $i = 1, \dots, e$. *This set is minimal.*

Proof. From the calculation of invariants, Lemma 1.3, and the equation (4) $s_i + kt_i = mu_i$, the monomial $U^{s_i}V^{t_i}$ is contained in the generating set of Lemma 1.3. It suffices to show that each monomial U^sV^t with $s + kt \equiv 0 \pmod{m}$, $0 \le s \le m$, $0 \le t \le m$ can be written as a product of the $U^{s_i}V^{t_i}$.

From (5) we have the partition $t_1 = 0 < t_2 = 1 < \cdots < t_e$, and in Lemma 2.5 below we show that $t_e = m$. Suppose $t = t_i$ for some i; then by (2) $0 \le s, s_i \le m$ so since the solution to $s_i + kt_i \equiv 0 \pmod{m}$ is unique modulo m (since $\gcd(k, m) = 1$), $s = s_i$.

Therefore we need only consider the case that $t_i < t < t_{i+1}$. Let s be the unique solution to $s + kt \equiv 0 \pmod{m}$ with $0 \le s < m$. We will prove:

Claim. $s \ge s_i$ (and thus $s > s_i$ since $t \ne t_i$).

Proof of the claim. Multiplying $s + kt = mu$ by u_i and $s_i + kt_i = mu_i$ by u, we subtract and obtain

$$su_i - s_iu + k(tu_i - t_iu) = 0.$$

The inequalities $tu_i - ut_i \le 0$ and $u_i \ge u$ from $t_i < t$ follow from Lemma 2.6 below. Therefore

$$0 \le k(ut_i - tu_i) = su_i - us_i \le u(s - s_i)$$

so $s \ge s_i$ as claimed. □

By induction on the exponent of V, $U^sV^t = (U^{s-s_i}V^{t-t_i})(U^{s_i}V^{t_i})$ can be written as a product of the claimed monomials, and each monomial is critical. Therefore the theorem follows from the claim and the two lemmas. □

Lemma 2.5. *t_{i+1} is the numerator of $[[b_2; a_3, \ldots, a_i]]$, $i = 2, \ldots, e-1$. Thus $t_e = m$ and $u_e = k$.*

Proof. Let

$$w_1 = -1, \quad w_2 = 0, \quad w_{i+1} = b_iw_i - w_{i-1}.$$

We then will prove that for any x that

$$[[b_2; b_3, \ldots, b_{i-1}, x]] = \frac{xt_i - t_{i-1}}{xw_i - w_{i-1}}, \quad i = 2, \ldots, e-1.$$

For $i = 2$ we have $[[x]] = x = x/1$. In general,

$$\begin{aligned}
[[b_2; b_3, \ldots, b_{i-1}, x]] &= [[b_2; b_3, \ldots, (b_{i-1} - 1/x)]] \\
&= \frac{(b_{i-1} - 1/x)t_{i-1} - t_{i-2}}{(b_{i-1} - 1/x)w_{i-1} - w_{i-2}} \\
&= \frac{x(b_{i-1}t_{i-1} - t_{i-2}) - t_{i-1}}{x(b_{i-1}w_{i-1} - w_{i-2}) - w_{i-1}} = \frac{xt_i - t_{i-1}}{xw_i - w_{i-1}}.
\end{aligned}$$

All we now need to show is that $\gcd(t_i, w_i) = 1$. This holds for $i = 2$ and in general because

$$t_{i+1}w_i - t_iw_{i+1} = (b_it_i - t_{i-1})w_i - t_i(b_iw_i - w_{i-1}) = t_{i-1}w_i + t_iw_{i-1}.$$

The last statment follows immediately using $s_e = 0$. □

Lemma 2.6. *Let (s,t,u) satisfy $s+kt=mu$ for $0 \le s < m$, $0 < t \le m$. Then u increases with t, and if*
$$\frac{t}{u} > \frac{t_i}{u_i}$$
for some i then $i < e$ and $t \ge t_{i+1}$.

Proof. Because $s+kt=mu$ we have
$$\frac{t}{u} = \frac{m}{k} - \frac{s}{ku} \le \frac{m}{k} = \frac{t_e}{u_e}$$
and thus $i<e$. Since the sequence t_i/u_i is monotonically increasing (cf. equation (5)), we can assume that $i<e$ is maximal with respect to $t/u > t_i/u_i$. Now u as a function of t is given by
$$u = \frac{s}{m} + \frac{kt}{m} = \left\lceil \frac{kt}{m} \right\rceil$$
since $0 \le s < m$ is uniquely determined by t. Therefore as t increases so does u, so if $t < t_{i+1}$ then $u \le u_{i+1}$. From this and (5) again, we have
$$\frac{s}{t} = \frac{st_i - ts_i}{tt_i} + \frac{s_i}{t_i} \ge \frac{1}{s_i s_{i+1}} + \frac{s_i}{s_{i+1}} = \frac{s_{i+1}}{t_{i+1}}.$$
By the maximality of i, equality must hold so $st_{i+1} = ts_{i+1}$; since $\gcd(s_{i+1}, t_{i+1}) = 1$ we have $s_{i+1} \mid s$ so $s \ge s_{i+1}$, which is a contradiction. □

We have done this in a purely algebraic way, motivated only by looking at the equations defining the generators themselves. In the next section, we relate this construction to the underlying toric structure and give it a geometric interpretation.

Remark. Note that of the elements $U^{s_i}V^{t_i}$, we have that $U^m, U^{m-k}V, V^m$ always occur. From the injection of semigroups we have the induced map
$$\begin{aligned} U_\sigma = \operatorname{Spec}(\mathbb{C}[U^{s_i}V^{t_i}]_{i=1}^e) &\to \operatorname{Spec}(\mathbb{C}[U^m, U^{m-k}V, V^m]) \\ &= \operatorname{Spec}(\mathbb{C}[A,C,B]/\langle C^m - A^{m-k}B\rangle). \end{aligned}$$
Since the quotient fields of each algebra agree and U_σ is normal, this allows us to compute the normalization of the latter using toric technology.

2.2. **Resolution of Singularities.** This section covers [6, §2.6]. Given any fan Δ, a refinement Δ' of Δ (i.e. each cone of Δ is a union of cones in Δ') defines a morphism $X(\Delta') \to X(\Delta)$ induced by the identity map of N. This map is *birational* and *proper* since it is an isomorphism of the open torus contained in each.

One can use fan refinement on a general singular toric variety to resolve its singularities, and this construction is particularly simple in the plane. For $N = \mathbb{Z}^2$ and σ a maximal cone in $N_\mathbb{R}$, the orbit of σ is a single point fixed by the torus T_N. Since U_σ is normal, the set of singular points has codimension ≥ 2 (viz. [18, Chapter II.5, Theorem 2]), therefore no other torus orbit may contain a singularity and this is the only possible singular point of U_σ.

In order to resolve this singularity, we put σ into standard form (generated by e_2, $me_1 - ke_2$, $0 \le k < m$ with $\gcd(m,k)=1$). If $m=1$ (so $k=0$) then the variety is nonsingular (and corresponds to $\mathbb{C}^2$); otherwise, we insert the ray e_1 (this is a blowup at the fixed point) since the cone generated by e_1, e_2 will be nonsingular and the lower cone will have a singular point which is "less" singular than the original one. To see this, we can position this smaller cone to standard form by rotating the lattice 90° (moving e_1 to e_2) and then translating the other vector vertically to put it in the position $(m_1, -k_1)$ with $m_1 = k$, $0 \leq k_1 < m_1$ and $k_1 = a_1 k - m$ for some integer $a_1 \geq 2$ as in Figure 3.

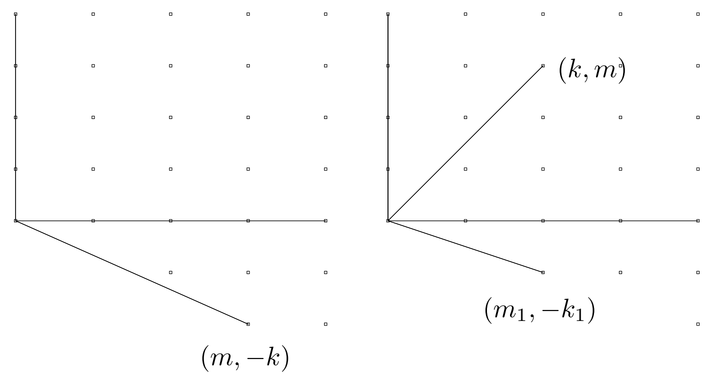


Figure 3. First resolution of the singularity and vertical translation

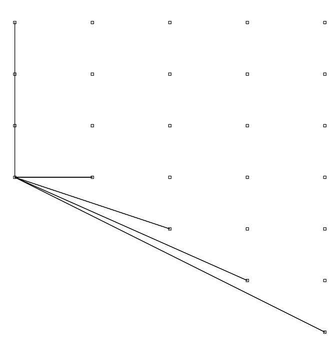
This corresponds to a smooth cone when $k_1 = 0$; otherwise

$$\frac{m}{k} = a_1 - \frac{k_1}{m_1} = a_1 - \cfrac{1}{\cfrac{m_1}{k_1}}$$

and the process can be repeated. We recognize this immediately as the Hirzebruch-Jung continued fraction for $m/k = [[a_1; a_2, \ldots, a_r]]$ as defined in the previous section. Fulton provides the following exercise [6, pp. 46–47]:

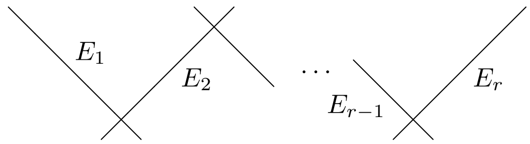
Proposition 2.7.

(a) *The rays inserted in the above process correspond exactly to the vertices on the edges of the boundary polygon obtained by the convex hull of the nonzero points in $\sigma \cap N$.*



(b) *There are r added vertices $v_1, \ldots, v_r$ between the given vertices $v_0 = e_2$, $v_{r+1} = me_1 - ke_2$, and $v_{i+1} = a_i v_i - v_{i-1}$.*

(c) *These added rays correspond to exceptional divisors $E_i \simeq \mathbb{P}^1$, forming a chain $E_1, \ldots, E_r$ having a linear dual graph*



with self-intersection numbers $(E_i \cdot E_i) = -a_i$*.*

(d) $\{v_0, \dots, v_{r+1}\}$ *is a minimal set of generators of the semigroup* $\sigma \cap N$*.*

(e) *If* Δ' *is the subdivision of* σ *obtained by the* v_i *then* $X(\Delta') \to U_{\sigma^\vee}$ *is the minimal equivariant resolution of singularities, i.e.* Δ' *is the coarsest nonsingular subdivision of* σ*.*

Proof. The key ingredient in this proof is understanding the lattice morphism obtained by rotation and vertical translation. This is the composition

$$\begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}.$$

If we begin with the vectors $v_0 = (0, 1)$ and $v_{r+1} = (m, -k)$, we find that $v_1 = (1, 0)$ is indeed the first vertex on the boundary of the convex hull. Now apply the change of basis, which takes $v_1 = (1, 0) \mapsto (0, 1)$ and $v_{r+1} = (m, -k) \mapsto v_r = (m_1, -k_1)$. Again, the image of v_2 will be the next vertex on the boundary, and this argument continues until we have

$$\begin{pmatrix} 0 & -1 \\ 1 & a_{e-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

i.e. $m_{r-1} = 1$ which coincides with the termination of the continued fraction. This proves (a) and the first statement in (b).

To prove the second statement in (b), we argue as follows: after applying the first automorphism, we have moved $v_0 \mapsto (-1, a_1)$, $v_1 \mapsto (0, 1)$, and $v_2 \mapsto (1, 0)$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ a_1 \end{pmatrix}.$$

This identity will also hold under any change of basis. In fact, at any given stage, after i automorphisms, we have the same setup, with the coordinates $v_{i-1} = (-1, a_i)$, $v_i = (0, 1)$ and $v_{i+1} = (1, 0)$ as claimed.

For (c), note that the process of adding a ray is equivalent to a blowup, as in [6, §1.1, p.6]; each v_i determines a curve $E_i \simeq \mathbb{P}^1$. Because this is indeed the desingularization, the rest follows as in [6, §2.5, p.44].

For (d), to prove that the vertices of the convex hull constitute the unique minimal basis of the semigroup $\sigma \cap N$, we note that any two neighboring support points form a basis of the additive group of lattice points because together with the origin these points bound a triangular region with no other lattice point in its closure (by definition) thanks to the two-dimensionality of our situation so the pair must have unimodular determinant. Therefore every vertex is necessarily a basis element of the semigroup as it cannot be obtained from the others, by construction.

We conclude by stating that Δ', the subdivision of σ by the v_i is a nonsingular fan, and this is minimal by construction (at each stage, we take the coarsest subdivision by drawing the ray e_1). □

What we have done is provide a method for a unimodular triangulation of any cone $\sigma \subset N$ in the plane. The procedure for computing a toric variety from such a cone involves computing the dual cone $\sigma^\vee$, for which the same procedure will apply. If we identify $M = N = \mathbb{Z}^2$, we know that the dual cone $\sigma^\vee$ is generated by $u_1 = (1, 0)$ and $u_2 = (k, m)$. To convert this to standard form, we apply the automorphism obtained from Lemma 1.1:

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & m \end{pmatrix} = \begin{pmatrix} 0 & m \\ 1 & -(m-k) \end{pmatrix}.$$

By part (d) of the previous proposition applied to $\sigma^\vee$, the Hilbert basis for S_σ is given by the continued fraction expansion of $m/(m-k) = [[b_2; b_3, \ldots, b_{e-1}]]$ obtained as the vertices occuring in the convex hull of nonzero lattice points in $\sigma^\vee$, where $u_1 = (0, 1)$, $u_2 = (1, 0)$, up to $u_e = (m, -(m-k))$ where $u_{i+1} = b_i u_i - u_{i-1}$ for $i = 2, \ldots, e-1$. So if we set $u_i = (p_i, q_i)$ then a minimal set of generators is

$$Y, X, \ldots, X^{p_i} Y^{q_i}, \ldots, X^m Y^{k-m}.$$

To put this in a more familiar form, we transform the cone back to its normal form via the inverse automorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and then make the substitution $X' = U^m$, $Y' = V/U^k$. By inspection, the first generator becomes $Y \mapsto X' = U^m$, the second $X \mapsto X'Y' = X^{m-k}Y$, and in general the same relation between generators (obtained from the b_i) will hold in the new basis. We have proven:

Proposition 2.8. *The algebra $A_\sigma = \mathbb{C}[S_\sigma]$ has Hilbert basis $U^{s_i} V^{t_i}$ for $i = 1, \ldots, e$ where the embedding dimension e and the exponents are defined as follows: let $m/(m-k) = [[b_2; b_3, \ldots, b_{e-1}]]$ be the Hirzebruch-Jung continued fraction expansion with $b_i \geq 2$. Then*

$$\begin{array}{llll} s_1 = m, & s_2 = m-k, & s_{i+1} = b_i s_i - s_{i-1}, & \multirow{2}{*}{$i = 2, \ldots, e-1.$} \\ t_1 = 0, & t_2 = 1, & t_{i+1} = b_i t_i - t_{i-1}, & \end{array}$$

As a nice application, we prove:

Proposition 2.9. *Let σ be generated by e_2 and $(k+1)e_1 - ke_2$. Then S_σ is the* rational double point of type A_k

$$A_\sigma = \mathbb{C}[A, B, C]/\langle C^{k+1} - AB \rangle.$$

The resolution of singularities has k exceptional divisors in a chain, each isomorphic to $\mathbb{P}^1$ and with self intersection -2.

Proof. By the preceding proposition, we have the fraction $(k+1)/\left((k+1)-k\right) = k+1 = b_2$. We calculate:

$$\begin{array}{lll} s_1 = k+1, & s_2 = (k+1)-k = 1, & s_3 = (k+1)(1) - (k+1) = 0 \\ t_1 = 0, & t_2 = 1, & t_3 = (k+1)(1) - 0 = k+1 \end{array}$$

so $A_\sigma = \mathbb{C}[U^{k+1}, UV, V^{k+1}]$ and clearly $(UV)^{k+1} = (U^{k+1})(V^{k+1})$ is the only syzygy so we indeed have the claimed variety.

In order to compute the resolution of singularities, we must triangulate the dual cone, which is generated by e_1^* and $ke_1^* + (k+1)e_2^*$, and thus we need the Hirzebruch-Jung continued fraction for $(k+1)/k$. The claim is that this is $(k+$

$1)/k = [[\underbrace{2; 2, \ldots, 2}_{k}]]$, and from Proposition 2.7 the rest follows. But the claim follows from $(1+1)/1 = 2$ and then by induction and

$$\frac{k+1}{k} = 2 - \frac{k-1}{k} = 2 - \cfrac{1}{\cfrac{k}{k-1}}.$$

□

The last exercise is:

Proposition 2.10. *Let σ be generated by e_2 and $me_1 - ke_2$ as above, and let σ' be generated by e_2 and $m'e_1 - k'e_2$ with $0 < k' < m'$, $\gcd(m', k') = 1$. Show that $U_{\sigma'} \simeq U_\sigma$ iff $m' = m$ and ($k' = k$ or $k'k \equiv 1 \pmod{m}$).*

Proof. From Proposition 1.4 and the proof of it, we can write $U_\sigma = \mathbb{C}^2/G_{m,k}$ and $U_{\sigma'} = \mathbb{C}^2/G_{m',k'}$. Now it is clear that these two varieties are isomorphic iff the groups $G_{m,k}$ and $G_{m',k'}$ are conjugate in $GL_2(\mathbb{C})$. It is therefore necessary that $m = m'$ (so the groups have the same order). Recall that $G_{m,k}$ is generated by $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^k \end{pmatrix}$ and likewise for $G_{m',k'}$. Now two scalar (diagonal) matrices are conjugate iff their diagonal elements are equal (up to permutation), and two cyclic subgroups are conjugate iff two generators are conjugate. Since we may assume one generator is of the above form (by taking a different primitive ζ), we must have $k = k'$ or we permute the diagonals by taking the power k' when $kk' \equiv 1 \pmod{m}$, and no other. This completes the proof.

Fulton [6, note 23, p.136] also suggests the following argument: if $k = k'$ it is clear (because the cones are the same) and otherwise we let $kk' = 1 - pm$ and we have the explicit automorphism

$$\begin{pmatrix} k & m \\ p & -k' \end{pmatrix} \begin{pmatrix} 0 & m \\ 1 & -k \end{pmatrix} = \begin{pmatrix} m & 0 \\ -k' & 1 \end{pmatrix}$$

of unimodular determinant which carries σ onto σ' which induces the isomorphism $U'_\sigma \to U_\sigma$. For the converse, we note that the above procedure for resolving the singularities of U_σ and $U_{\sigma'}$ is minimal and therefore the dual graph of the exceptional divisor is thereby uniquely determined and so too the sequence of numbers $a_1, \ldots, a_r$ up to replacing it by the reverse sequence $b_1, \ldots, a_r$ (which has the same dual graph). Since the continued fraction expansion is unique, we need only show $m/k = [[a_1; a_2 \ldots, a_r]]$ implies $m/k' = [[a_r; a_{r-2}, \ldots, a_1]]$ where $kk' \equiv 1 \pmod{m}$, and this follows from the above automorphism because it maps $v_1 = (0,1) \mapsto (m, -k')$ and $v_e = (m,k) \mapsto (0,1)$ and therefore we will calculate the vertices in the reverse order. □

Remark. In fact, any two-dimensional analytic space (not necessarily coming from a toric variety) has a resolution of singularities for which blowups and continued fractions are the main ingredients (see [10, Chapter 2] for the proof and many examples).

2.3. **Equations.** One might expect that because they come from dual objects, the numbers $a_1, \ldots, a_r$ and $b_2, \ldots, b_{e-1}$ would be related. Indeed, they are:

Proposition 2.11 (Riemenschneider [13, 14])**.** *Let*

$$\begin{aligned} m/k &= [[a_1; a_2, \dots, a_r]], \\ m_1/k_1 &= [[a_2; a_3, \dots, a_r]], \\ m_1/(m_1 - k_1) &= [[b_2; b_3, \dots, b_{e-1}]]. \end{aligned}$$

Then

$$\frac{m}{m-k} = [[\underbrace{2; 2, \dots, 2}_{a_1}, a_2 + 1, a_3, \dots, a_r]].$$

This proposition gives an inductive procedure for calculating the b_i from the a_j and vice versa—at each stage we convert a_j to a string of 2s of length a_j as appropriate and continue. In other words:

Algorithm 2.12 (Calculating b_i from a_j). Given $a_1, \dots, a_r$ from m/k, we form r rows of $a_j - 1$ points as follows:

$$\begin{array}{llll} \underbrace{\times \ \dots \ \times}_{a_1 - 1} & & & \\ & \underbrace{\times \ \dots \ \times}_{a_2 - 1} & & \\ & & \vdots & \\ & & & \underbrace{\times \ \dots \ \times}_{a_r - 1} \end{array}$$

Then the number of points in column i is equal to $b_i - 1$.

Example 2.13. If we take $a_1, \dots, a_5 = 5, 2, 2, 3, 2$ with $r = 5$, and then draw

$$\begin{array}{ccccc} \times & \times & \times & \times & \\ & & & \times & \\ & & & \times & \\ & & & \times & \times \\ & & & & \times \end{array}$$

so $b_2, \dots, b_6 = 2, 2, 2, 5, 3$. Indeed,

$$5 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{2}}}} = [[5; 2, 2, 3, 2]] = \frac{47}{11}$$

and

$$2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{5 - \cfrac{1}{3}}}} = [[2; 2, 2, 5, 3]] = \frac{47}{36} = \frac{47}{47 - 11}.$$

Remark. The duality of the a_j and b_i by this point diagram gives

$$[[a_1; a_2, \dots, a_r, 1, b_{e-1}, \dots, b_2]] = 0.$$

In fact, the set K_{e-2} of continued fractions $[[k_2; k_3, \dots, k_{e-1}]]$ that represent zero is the Catalan number $C_{e-2} = \frac{1}{e-2}\binom{2(e-3)}{e-3}$. For more on continued fractions which represent zero and their relationship to versal deformation of cyclic quotient singularities and their P-resolutions, see [20].

One easy consequence of the construction is the following:

Proposition 2.14. *With $m/k = [[a_1; \dots, a_r]]$ and $m/(m-k) = [[b_2; \dots, b_{e-1}]]$, we have*

$$\sum_{j=1}^{r}(a_j - 1) = \sum_{i=2}^{e-1}(b_i - 1)$$

so that

$$e = 3 + \sum_{j=1}^{r}(a_j - 1).$$

This allows us to calculate the embedding dimension e directly from the continued fraction for m/k.

Remark. Oda [11] makes the following remark: Let $E_1, \dots, E_r$ be the exceptional curve for the minimal resolution for the singularity of U_σ at the origin. Since $(E_j \cdot E_j) = -a_j$, $(E_j \cdot E_{j+1}) = 1$, and $(E_j \cdot E_i) = 0$ for nonadjacent E_i, E_j, we get

$$-\left((E_1 + \cdots + E_r) \cdot (E_1 + \cdots + E_r)\right) = 2 + \sum_{i=1}^{r}(a_j - 2) = e + 1$$

which is also the multiplicity of U_σ at the singularity. This is also the volume of the polygon formed by the difference of the cone and the convex hull of the minimal generators (since each of these triangles has area 1/2).

Using the minimal generating set for the variety, we can in the usual way take $\mathbb{C}[X_1, \dots, X_e] \to U_\sigma$, $X_i = U^{s_i}V^{t_i}$, and let I be the kernel of this map, a binomial ideal defining the relations among the X_i. We can find this ideal directly:

Proposition 2.15 (Riemenschneider [13, 14])**.** *I is generated by $X_iX_j - Y_{ij}$ for $2 \le i+1 \le j-1 \le e-1$ where*

$$Y_{ij} = \begin{cases} X_{i+1}^{b_{i+1}}, & i+1 = j-1, \\ X_{i+1}^{b_{i+1}-1}X_{i+2}^{b_{i+2}-2}\dots X_{j-2}^{b_{j-2}-2}X_{e-1}^{b_{e-1}-1}, & i+1 < j-1. \end{cases}$$

Moreover, this set of generators is minimal.

We can also write this as a *quasideterminantal* variety (cf. [15, 16])

$$\begin{pmatrix} X_1 & & X_2 & \dots & X_{e-2} & & X_{e-1} \\ X_2 & & X_3 & \dots & X_{e-1} & & X_e \\ & X_2^{b_2-2} & & \dots & & X_{e-1}^{b_{e-1}-2} & \end{pmatrix}.$$

The generalized minors of a quasideterminantal

$$\begin{pmatrix} A_1 & & A_2 & \dots & A_{\ell-1} & & A_\ell \\ B_1 & & B_2 & \dots & B_{\ell-1} & & B_\ell \\ & C_{1,2} & & \dots & & C_{\ell-1,\ell} & \end{pmatrix}$$

are given by $A_iB_j - B_i(C_{i,i+1}\dots C_{j-1,j})A_j$ for $1 \le i < j \le \ell$. (If we let $C_{i,i+1} = 1$ then we see that this is indeed a generalization of the ordinary minors.)

Example 2.16. If $m = 5$, $k = 3$, we have $5/3 = [[2;3]]$ so we form

$$\begin{pmatrix} A & & B & & D \\ B & & C & & D \\ & 1 & & C & \end{pmatrix}$$

and therefore we have the representation

$$\mathbb{C}[A,B,C,D]/\langle AC - B^2, AD - BC^2, BD - C^3\rangle.$$

From Example 2.1, we have $A_\sigma = \mathbb{C}[U^5, U^2V, UV^3, V^5]$ so we can verify this calculation in Macaulay:

```
i1 : R = QQ[U,V, A,B,C,D, MonomialOrder => Eliminate 2];

o1 : PolynomialRing

i2 : I = ideal( D-V^5, C-U^2*V, B-U*V^3, A-V^5 );

o2 : Ideal of R

i3 : selectInSubring(1,gens gb I)

o3 = {0} | B^2-AC BC^2-AD C^3-BD |

             1       3
o3 : Matrix R  <--- R
```

As one final application of this, we can prove the following:

Claim. The two-dimensional toric variety U_σ is a complete intersection iff it is of the form given in Proposition 2.9, i.e. it is a rational double point $\mathbb{C}[A,B,C]/\langle C^{k+1} - AB\rangle$.

Proof. We embed $U_\sigma \hookrightarrow \mathbb{C}[X_1, \ldots, X_e]$; the ideal is generated by $f = (1/2)(e-1)(e-2)$ elements, and thus U_σ is a complete intersection iff $e = f + \dim U_\sigma = f + 2$ iff $(1/2)(e-2)(e-3) = 0$ iff $e = 3$ (since $e = 2$ implies the variety is $\mathbb{C}^2$) iff $m - k = 1$ (the algorithm stops after 2 steps) iff $m = k + 1$. □

2.4. **Implementation.** The following Maple routines can be used to compute finite Hirzebruch-Jung fractions:

```
with(numtheory,cfrac):
# Find the Hirzebruch-Jung continued fraction of a/b
hjcfrac := proc (q) local m,k,l,a,cf;
  m := numer(q):
  k := denom(q):
  cf := [];
  while ( k>1 ) do
    a := floor(m/k)+1;
    l := a*k-m;
    m := k;
    k := l;
    cf := [op(cf),a];
  od;
  cf := [op(cf),m];
end:
```

```
# Pretty-print the H-J fraction
hjcprint := proc (cf) local s,i;
  i := nops(cf);
  s := convert(cf[i],symbol);
  i := i-1;
  while ( i>0 ) do
    s := cf[i]-1/s;
    i := i-1;
  od;
  print(s);
end:

# User function
hjc := proc (q);
  hjcprint(hjcfrac(q));
end:

# Evaluate an expanded fraction
hjceval := proc (cf) local q,i;
  i := nops(cf);
  q := cf[i];
  i := i-1;
  while ( i>0 ) do
    q := cf[i]-1/q;
    i := i-1;
  od;
  q;
end
```

For example, the command `hjcfrac(47/11)` returns $[5, 2, 2, 3, 2]$, the command `hjceval([3,2])` returns $5/2$, and the command `hjc(87/23)` prints the continued fraction $[[4; 5, 3, 2]]$ in expanded form.

Using this, we can easily compute A_σ:

```
# Computes the generators for S_sigma defined
# by a cone in standard form in U,V coords
mingensstd := proc (A) local G;
  G := coord(mingensstdeng(A,1));
end:

# Computes the standardized generators for any cone sigma
mingens := proc (A) local T,B,G;
  T,B := stdform(A);
  G := coord(mingensstdeng(col(B,2),1));
end:

# Convert coordinates to monomials
coord := proc (G) local H,i;
  H := [];
  for i from 1 to nops(G) do
    H := [ op(H), U^(G[i][1])*V^(G[i][2]) ];
  od;
  H;
end:
```

```
# Computes the convex hull of the cone sigma
convhull := proc (A) local T,B,G,i;
  T,B := stdform(A);
  G := mingensstdeng(col(B,2),2);
  T := inverse(T);
  for i from 1 to nops(G) do
    G[i] := multiply(T,G[i]);
  od;
  G;
end:

# Engine: 1 means in U,V coords, 2 means in X,Y coords
mingensstdeng := proc (A,flg) local m,k,cf,s,t,i,gens;
  m := A[1]; k := -A[2];
  if ( flg = 1 ) then
    s := [m,m-k,0];
    t := [0,1,0];
    cf := hjcfrac(m/(m-k));
  else
    s := [0,1,0];
    t := [1,0,0];
    cf := hjcfrac(m/k);
  fi;
  gens := [ [s[1],t[1]], [s[2],t[2]] ];
  i := 1;
  for i from 1 to nops(cf) do
    s[3] := cf[i]*s[2]-s[1];
    t[3] := cf[i]*t[2]-t[1];
    gens := [ op(gens), [s[3],t[3]] ];
    s := [op(2..3,s),0];
    t := [op(2..3,t),0];
  od;
  gens;
end:
```

For example, `mingensstd([5,3])` returns $[U^5, U^2V, UV^3, V^5]$ whereas the command `mingens([[-13,11],[10,7]])` gives the highly nontrivial

$$[U^{201}, U^{61}V, U^{43}V^4, U^{25}V^7, U^7V^{10}, U^3V^{33}, U^2V^{89}, UV^{145}, V^{201}]$$

We also have a convex hull function: `convhull(array([[1,3],[0,-2]]))` for the cone defined by $(1,0),(3,-2)$ gives $[[1,0],[2,-1],[3,-2]]$.

3. Other topics

Continued fractions pop up in other ways in toric geometry. We conclude with a few examples.

First, we consider the plane curve $C : X^s = Y^r$ $(r > s \geq 1)$. C has a *cusp singularity* at the origin whenever $r \nmid s$ and $s \nmid r$, e.g. $X^3 = Y^2$. C is parameterized by $T \mapsto (T^r, T^s)$, which we recognize as a one-parameter subgroup λ_α where $\alpha = re_1 + se_2$ coming from the complex torus T_N where $N = \mathbb{Z}^2$. Desingularizing the curve C can be done by refining the cone generated by e_1, e_2 (which is nonsingular and therefore corresponds to blowing up the plane at the origin).

We have the following theorem:

Theorem 3.1 (Hirzebruch, Jurkiewicz [9]). *Let*

$$\frac{r}{s} = a_m + \cfrac{1}{a_{m-1} + \cfrac{1}{\cdots + \cfrac{1}{a_0}}} = ((a_m; a_{m-1}, \dots, a_0))$$

$a_0 \geq 2$, $a_1, \dots, a_m \geq 1$ *be the* ordinary *continued fraction. Then there exists a sequence of blowups* $X_s \to X_{s-1} \to \cdots \to X_0 = \mathbb{C}^2$ *such that the proper inverse image* C' *of* C *is the affine line and the exceptional curve (over the origin) consists of a chain of projective lines intersecting transversally with self intersection numbers obtained from the* a_i.

Example 3.2. For $X^3 = Y^2$, we have $r = 3$, $s = 2$, and $3/2 = 1 + 1/2$. A single blowup of the plane at the origin will resolve the singularity, and the dual graph is a single $\mathbb{P}^1$ with self-intersection number -2.

Cusp singularities also occur in considering the graded ring of Hilbert modular forms for a real quadratic extension of $\mathbb{Q}$, and this was the case taken up by Hirzebruch. This has also been generalized to higher-dimensional analogues of periodic continued fractions by Tsuchihashi (Oda [11] and the paper by Cohn [5] give nice introductions to this area).

One is naturally led after calculating quotient surface singularities to consider their infinitessimal deformations. Riemenschneider [13] began this analysis, and it was generalized by Pinkham [12] for cyclic quotients and by Altmann [1] in reference to partial and maximal resolutions with particular attention to the toric application.

Finally, the rational cubic curve is given by the graded algebra

$$\mathbb{C}[T, T^2, T^3] = \mathbb{C}[A, B, C]/\langle B - A^2, C - A^3 \rangle.$$

Notice that this ring is isomorphic to the polynomial ring in one variable, and in particular it has exactly one generator of any given degree. This is the simplest example of an $\mathcal{A}$-*graded algebra*, an $\mathbb{N}^d$-graded k-algebra R such that $\dim_k(R_b) = 1$ or 0 according to when $b \in \mathbb{N}^d$ is an element of the semigroup generated by a finite set $\mathcal{A} \subset \mathbb{N}^d$ (as defined in [22]). The unigraded case exhibited above was investigated by Arnold [2], and the following fact was found:

Proposition 3.3. *Let*

$$\frac{v}{u} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_r}}}$$

be the standard *continued fraction expansion of* v/u *(note the plusses). The number of (infinite-dimensional)* $\mathcal{A}$*-graded algebras up to isomorphism with three multiplicative generators of degrees* $1, u, v$ *where* $d = 1$ *and* $\mathcal{A} = \{1\}$ *(i.e. exactly one monomial of each degree) is given by the number* $2\left(\sum_{i=1}^{r} a_i\right) + 1$.

Any A-graded algebra with two multiplicative generators is uniquely determined by the degree of the second generator.

Indeed, for multiplicative generators of degrees $1,2,3$ as above we find that

$$\frac{3}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$$

so there are exactly $2(1+1)+1 = 5$ different $\mathcal{A}$-graded algebras other than the original ideal defining the cubic curve; they are given by quotients of $k[A,B,C]$ by the ideals:

$$\langle A^2, AB, B^2\rangle,\ \langle A^2, AB, B^2 - AC\rangle,\ \langle A^2, AB, AC, B^3\rangle,$$
$$\langle A^2, AB, AC, B^3 - C^2\rangle,\ \langle A^2, AB, AC, C^2\rangle.$$

If we restrict to k-algebras defined by monomial ideals, a slight modification of the argument shows that the count of k-algebras is given by $(\sum_{i=1}^{r} a_i) + 1$. This provides a way to count the number of initial ideals of a toric variety given by three integer points $1, u, v \in \mathbb{N}$: we need only add in the cases where the generators of degree u or v (or both) are zero (all of which are uniquely determined), totalling $(a_1 + \cdots + a_r) + 4$. Unfortunately, this nice result does not extend to the case of four generators, as one can find in [22].

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