ERRATA AND ADDENDA: *QUATERNION ALGEBRAS*

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Errata

Mathematical glitches and errors.

(1) (2.4.3): the second equality only holds for $n = 2$, so replace with "where
$$\mathrm{SU}(n) := \{A \in \mathrm{SL}_n(\mathbb{C}) : A^* = A^{-1}\}$$
and $A^* = \overline{A}^{\mathsf{t}}$ is the (complex) conjugate transpose of A."

(2) Proof of Lemma 3.4.2, line 5: "B" should be "K".

(3) Section 4.1, page 48, line 1, "$\det f$" should be "$\det f = 1$".

(4) 4.2.13: "$\mathrm{nrd}(a\alpha) = a^2\alpha$" should be "$\mathrm{nrd}(a\alpha) = a^2 \,\mathrm{nrd}(\alpha)$".

(5) 4.2.20: should be $Q'(x')$ and $T'(x', y')$.

(6) 4.2.21: replace with "$\langle a_1, \dots, a_n\rangle := \langle a_1\rangle \boxplus \dots \boxplus \langle a_n\rangle$" (swap sides).

(7) 4.5.8: "Then there is a" should be "Writing $V = B^0$, there is a"; replace $\mathrm{trd}(B)$ with $\mathrm{trd}(v)$"; and in (4.5.9) replace "B^0" by "V".

(8) Example 4.5.13: "$\det^0$" should be "$\det|_{B^0}$".

(9) Example 4.5.14: "det" should be "nrd".

(10) Proof of Proposition 4.5.17: "$x\overline{x}\,\overline{x}^{-1}$" should be "$x\overline{v}\,\overline{x}^{-1}$".

(11) Beginning of section 5.3: we have only defined quadratic forms when $\mathrm{char}\, F \neq 2$. Replace "we pause our assumption and allow F of arbitrary characteristic" with "the reader may continue to suppose that $\mathrm{char}\, F \neq 2$, but the constructions in this section work quite generally, so the reader may also wish to return to this section after reading Chapter 6 and allow $\mathrm{char}\, F = 2$."

(12) Proof of Lemma 5.3.14: the one given only works when $\mathrm{char}\, F \neq 2$. Add "We give a proof when $\mathrm{char}\, F \neq 2$; for another approach that works more generally, see Exercise 5.20" to the start of the proof, and remove the parenthetical clause "(Alternatively, this can be viewed as a graded tensor product; see Exercise 5.21)."

(13) Proof of Lemma 5.4.2: "$y \leftarrow y - 2x/T(y, y)$" should be "$y \leftarrow y - T(y, y)x/2$". Better to elaborate on the computation: "Then replacing $y \leftarrow y - Q(y)x = y - T(y, y)x/2$ gives y

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isotropic, since

$$Q(y - Q(y)x) = Q(y) + Q(Q(y)x) + T(y, -Q(y)x) = Q(y) - Q(y) = 0.\text{''}$$

(14) Proof of Lemma 5.4.7: "$\gamma = \alpha\beta^{-1}$" should be "$\gamma = -\alpha\beta^{-1}$".
(15) 5.6.7: "δ" should be "ζ".
(16) (5.6.10): "$-\operatorname{nrd}(v)$" should be "$Q(v)$".
(17) Proof of Theorem 6.4.7: $Q \sim \langle 1 \rangle$ should be $Q \simeq \langle 1 \rangle$.
(18) Proof of Theorem 7.1.2: replace "; in either case, $I\alpha \subseteq I$ and I is a right ideal as well" with "We cannot have $I \cap I\alpha = \{0\}$ since then $6 = \dim(I + I\alpha) < 4$, impossible. Thus $I\alpha = I$ and I is a right ideal as well."
(19) Proof of Theorem 7.1.5: "$\beta \in B \otimes_F F^{\text{sep}} \simeq \mathrm{GL}_2(F^{\text{sep}})$" should be "$\beta \in (B \otimes_F F^{\text{sep}})^\times \simeq \mathrm{GL}_2(F^{\text{sep}})$".
(20) Proof of Theorem 7.3.5(b): Replace with "For (b), let $W \subseteq V$ be a submodule of the semisimple B-module V. Among all injective maps from W into a finite direct sum of simple B-modules (a nonempty collection from $W \subseteq V$), let $\phi\colon W \to \sum_i V_i$ have the minimal number of simple factors. We claim that ϕ is an isomorphism. Indeed, for each j, composing with the projection gives a map $\phi_j\colon W \to \bigoplus_{i \neq j} V_i$ with fewer factors, hence by minimality it is not injective; thus there exists $w_j \in W$ nonzero such that $\phi(w_j) \in V_j$, and since V_j is simple we get $\phi(Bw_j) = V_j$. Putting these together for all j, we conclude that ϕ is surjective. For the second statement on quotient modules, suppose $\phi\colon V \to Z$ is a surjective B-module homomorphism; then $\phi^{-1}(Z) \subseteq V$ is a B-submodule, and $\phi^{-1}(Z) = \sum_i W_i$ is a sum of simple B-modules, and hence by Schur's lemma $Z = \sum_i \phi(W_i)$ is semisimple."
(21) Corollary 7.7.11: should assume B is a division algebra. Replace the statement with "Let B be a central division F-algebra and let K be a maximal subfield. Then $\dim_F B = (\dim_F K)^2$." And the proof should read "Since B is a division algebra and K is maximal subfield, in fact K is a maximal commutative F-subalgebra, so $C_B(K) = K$ and thus by Proposition 7.7.8(b) we have $\dim_F B = (\dim_F K)^2$."
(22) 9.2.1: should be $f\colon M \to P$ (missing f).
(23) Proof of Lemma 9.4.6: replace $(M : x)$ by $\mathfrak{a}$ (since this is not the whole colon ideal), and replace "Then $\mathfrak{a}$ is an ideal of R" by "Then $\mathfrak{a}$ is an ideal of R, nonzero by Lemma 9.3.5(a)".
(24) Proof of Proposition 9.4.7: replace $\bigcap_{\mathfrak{p}}$ with $\bigcap_{\mathfrak{m}}$.
(25) Paragraph before Theorem 9.4.9: clarify by replacing with the following.

To conclude this section, suppose that R is a Dedekind domain. We characterize in a simple way the conditions under which a collection $(M_{(\mathfrak{p})})_{\mathfrak{p}}$ of $R_{(\mathfrak{p})}$-lattices arise from a global R-lattice. Recall that a fractional ideal of R can be factored uniquely into a product of prime ideals, and hence by the data of these primes and their exponents. So as in 9.4.5, localization furnishes a bijection between fractional R-ideals $\mathfrak{a} \subseteq F$ and collections of fractional $R_{(\mathfrak{p})}$-ideals $(\mathfrak{a}_{(\mathfrak{p})})_{\mathfrak{p}}$ indexed by the primes $\mathfrak{p}$ satisfying $\mathfrak{a}_{(\mathfrak{p})} = R_{(\mathfrak{p})}$ for all but finitely many primes $\mathfrak{p}$; an inverse is $(\mathfrak{a}_{(\mathfrak{p})})_{\mathfrak{p}} \mapsto \bigcap_{\mathfrak{p}} \mathfrak{a}_{(\mathfrak{p})}$. So too can a lattice be understood by a finite number of localized lattices, once a "reference" lattice has been chosen (to specify the local behavior of the lattice at other primes).

(26) Proof of Theorem 9.4.9: in the last paragraph, it is a bit more complicated due to the infinite intersection. Instead, replace the last paragraph of the proof with the following.

By Lemma 9.4.6, the association $(N_{(\mathfrak{p})})_{\mathfrak{p}} \mapsto \bigcap_{\mathfrak{p}} N_{(\mathfrak{p})}$ is a left inverse to $N \mapsto (N_{(\mathfrak{p})})_{\mathfrak{p}}$. Conversely, given a collection $(N_{(\mathfrak{p})})_{\mathfrak{p}}$ and letting $N' = \bigcap_{\mathfrak{p}} N_{(\mathfrak{p})}$, we claim that $N'_{(\mathfrak{p})} = N_{(\mathfrak{p})}$ for all $\mathfrak{p}$ (providing a right inverse). Indeed, the inclusion $(\subseteq)$ is immediate, so we prove $(\supseteq)$. For all but finitely many $\mathfrak{p}$ we have $N'_{(\mathfrak{p})} = M_{(\mathfrak{p})} = N_{(\mathfrak{p})}$, the first as observed above and the second by construction. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the finitely many remaining primes, and let $x \in N_{(\mathfrak{p}_i)}$. As in the proof of Lemma 9.4.6, consider the nonzero ideal

$$\mathfrak{a} := \{a \in R : ax \in N_{(\mathfrak{p}_j)} \text{ for all } j\}.$$

Then

$$\mathfrak{a}_{(\mathfrak{p}_j)} = \{a \in R_{(\mathfrak{p}_j)} : ax \in N_{(\mathfrak{p}_j)}\}$$

for all j and $\mathfrak{a} = \bigcap_{\mathfrak{p}} \mathfrak{a}_{(\mathfrak{p})}$ as above. Since $x \in N_{(\mathfrak{p}_i)}$, we have $\mathfrak{a}_{(\mathfrak{p}_i)} = R_{(\mathfrak{p}_i)}$, so there exists $a \in \mathfrak{a} \smallsetminus \mathfrak{p}_i$ (since $\mathfrak{a} \neq \mathfrak{p}_i\mathfrak{a}$ by unique factorization), and therefore $ax \in N'$ has $ax \in N'_{(\mathfrak{p}_i)}$, thus $x \in N'_{(\mathfrak{p}_i)}$.

(27) At start of section 10.2: "Throughout, let R be a noetherian domain".
(28) Below (11.2.8): replace with "$s = -\omega^2 = (1+i+j+k)/2$, and $t = (1+i-j+k)/2$".
(29) Proof of Proposition 11.3.4: replace "$\alpha = \mu\beta + \rho$" with "$\alpha = \beta\mu + \rho$".
(30) Remark 11.4.11: FGS should be [FGS2016].
(31) 11.5.7: the vertices given are for the dodecahedron.
(32) Proof of Theorem 14.3.4, below (14.3.5), "let $p \mid b$ be a prime divisor": delete.
(33) Proof of Lemma 14.7.5: rewrite the proof with the following.

We suppose that char $F_v \neq 2$ and leave the other case as an exercise (Exercise 14.23). If $-n_v \notin F_v^{\times 2}$, then we can take $t_v = 0$; this treats the case where v is a real place.

So suppose $-n_v \in F_v^{\times 2}$. Let π_v be a uniformizer and let $e_v \in R_v^\times$ be a nonsquare in $k_v^\times$ where k_v is the residue field. Returning to the Hilbert symbol (section 12.4), since

$$(-1, e_v)_v(-1, \pi_v)_v(-1, e_v\pi_v)_v = (-1, 1)_v = 1$$

and each of $e_v, \pi_v, e_v\pi_v \notin F_v^{\times 2}$, there exists $d_v \in F_v^\times \smallsetminus F_v^{\times 2}$ such that $(-1, d_v)_v = 1$. Then the Hilbert equation $-x_v^2 + d_v y_v^2 = 1$ has a solution $x_v, y_v \in F_v$; since $-4n_v \in F_v^{\times 2}$, rescaling (and substituting) gives instead $-x_v^2 + d_v y_v^2 = -4n_v$. Let $t_v := x_v$. Then $x^2 - t_v x + n_v$ has discriminant $t_v^2 - 4n_v = d_v y_v^2 \in F_v^\times \smallsetminus F_v^{\times 2}$ and so is separable and irreducible.

For the second statement we recall Lemma 13.2.1. In the field $K_v := F_v(\alpha)$, where α is a root of $x^2 - t_v x + n_v$, since $n_v = \operatorname{nrd}(\alpha) \in R$ we conclude α is in the valuation ring S of K; but then α is integral, so $t_v \in R$ as well. (One can also prove this statement directly.)

(34) Example 15.5.7: replace "$qp = -pq = 1$" with "$\left(\dfrac{-p}{q}\right) = -\left(\dfrac{q}{p}\right) = 1$".
(35) Proposition 15.6.7: all occurrences of $\mathfrak{a}$ should be replaced by R. The whole point of taking trace duals is to have $\operatorname{trd}(\alpha\beta) \in R$ for $\alpha \in I$ and $\beta \in I^\sharp$!
(36) §16.1, "and the product of two (say) right $\mathcal{O}$-ideals need not be again a right $\mathcal{O}$-ideal! To address this, for lattices I, J": in any ring A, the product of two right A-ideals is again an A-ideal! (There is a problem with the product of two *locally principal* right $\mathcal{O}$-ideals from being again locally principal, but it is too soon to say that. We also have that the product of a right $\mathcal{O}$-ideal and a left $\mathcal{O}$-ideal need not be left or right $\mathcal{O}$-ideal.) Replace with "To study ideals of $\mathcal{O}$ we must distinguish between left or right ideals and take care with products. For lattices I, J".
(37) 16.4.6 through (16.4.10): rewrite as follows.

16.4.6. For a nonzero ideal $\mathfrak{a}$ of R, we define the **absolute norm** (or **counting norm**) $\mathsf{N}(\mathfrak{a})$ to be

$$\mathsf{N}(\mathfrak{a}) := \#(R/\mathfrak{a}) < \infty. \tag{16.4.7}$$

We extend this definition multiplicatively to fractional ideals and to elements $a \in F^\times$ by defining $\mathsf{N}(a) := \mathsf{N}(aR)$. Then

$$\mathsf{N}(\mathfrak{a}) = |\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})|.$$

16.4.8. Similarly, if $I \subseteq B$ is a locally principal R-lattice, we define the **absolute norm** of I to be

$$\mathsf{N}(I) := \mathsf{N}([\mathcal{O}_{\mathsf{L}}(I) : I]_R) = \mathsf{N}([\mathcal{O}_{\mathsf{R}}(I) : I]_R), \tag{16.4.9}$$

the latter equality by Proposition 16.4.3. If I is integral then

$$\mathsf{N}(I) = \#(\mathcal{O}_{\mathsf{L}}(I)/I) = \#(\mathcal{O}_{\mathsf{R}}(I)/I).$$

By Proposition 16.4.3, we have

$$\mathsf{N}(I) = \mathsf{N}(\mathrm{N}_{B|F}(I));$$

and if B is simple with $\dim_F B = n^2$ then

$$\mathsf{N}(I) = \mathsf{N}(\mathrm{N}_{B|F}(I)) = \mathsf{N}(\mathrm{nrd}(I))^n. \tag{16.4.10}$$

(38) Example 16.5.12: the last equality in (16.5.14) and the final equality is wrong, since $1/p$ is not in any order! Should be $\mathbb{Z} + \dfrac{1}{p}\mathcal{O}^0$.
(39) Remark 16.5.19: confusion with d versus d_K, should read "as abelian groups, we have

$$\mathfrak{f} = f\mathbb{Z} + \sqrt{d}\mathbb{Z} = f \cdot S(d_K),$$

so $\mathfrak{f}$ is principal and hence certainly invertible as an ideal of $S(d_K)$—but not as an ideal of the smaller order $S(d)$."
(40) Proof of Proposition 16.4.3: in the first paragraph, better to reference Lemma 9.6.3.
(41) Proof of Main Theorem 16.6.1: for the application to Proposition 16.6.15(a), we need $I^3 = I^4$, which here reads $I^{n-1} = I^n$. This is obtained by taking $\alpha_1 = 1$, which can be justified as follows: We may suppose without loss of generality that $\alpha_1 = 1$: indeed, if $\mathfrak{p}$ is the maximal ideal of R and $k := R/\mathfrak{p}$ its residue field, then $I/\mathfrak{p}I \simeq k^n$ is a k-vector space with $1 \neq 0$, so we can extend to a basis and this lifts to a basis over R, by Nakayama's lemma. In the rest of the proof, replace n by $n-1$.
(42) Lemma 17.3.3: for (iii), also require "If further I, J are invertible with $\mathcal{O}_{\mathsf{R}}(I) = \mathcal{O}_{\mathsf{R}}(J)$".
(43) Proof of Lemma 17.4.6: "$\mathcal{O}'_{\mathfrak{p}} = \mathcal{O}_{\mathsf{R}}(J_{\mathfrak{p}})$" (replace I by J).
(44) Example 17.6.3: the matrix β should be $\begin{pmatrix} 1-b_0 & 0 \\ 0 & b_0 \end{pmatrix}$.
(45) Lemma 17.7.26: add "I" to the statement and change the proof to read: "For such $I \subseteq \mathcal{O}$, we have $\mathsf{N}(I) = [\mathcal{O} : I]_{\mathbb{Z}} \leq C$, the index taken as abelian groups. But there are only finitely many subgroups of $\mathcal{O}$ of index $\leq C$, since $\mathcal{O}$ is finitely generated: they correspond to the possible kernels of surjective group homomorphisms $\mathcal{O} \to A$ where $\#A = n \leq C$."
(46) Lemmas 18.1.1 and 18.2.8: say "nonzero". So "every nonzero two-sided $\mathcal{O}$-ideal contains a product of prime nonzero two-sided $\mathcal{O}$-ideals" and "Every nonzero two-sided ideal of $\mathcal{O}$ contains a (finite) product of prime nonzero two-sided ideals."
(47) After (19.1.1): replace sentence with "The set $\mathrm{Cl}(d)$ of $\mathrm{SL}_2(\mathbb{Z})$-classes of forms in $\mathcal{Q}(d)$ is finite, by reduction theory: when $d < 0$, every form in $\mathcal{Q}(d)$ is equivalent under the action of $\mathrm{SL}_2(\mathbb{Z})$ to a unique **reduced** form, of which there are only finitely many (see section 35.2)."
(48) (19.1.3): replace sentence with: "Conversely, the quadratic form is recovered from the norm form on $K = \mathbb{Q}(\sqrt{d})$ via

$$\begin{aligned} \mathrm{Nm}_{\mathfrak{a}} \colon \mathfrak{a} &\to \mathbb{Z} \\ \mathrm{Nm}_{\mathfrak{a}}(\alpha) &= \mathrm{Nm}_{K|\mathbb{Q}}(\alpha)/a \end{aligned}$$

where $a = \mathrm{Nm}(\mathfrak{a}) > 0$, with respect to an oriented basis."

(49) Proposition 19.4.1: add as the first sentence of the proof: "Multiplication is defined by 16.5.3."
(50) Proof of Theorem 20.3.3, before (20.3.5): "to show that I is left invertible" should be "to show that $I^{-1}I = \mathcal{O}_{\mathsf{R}}(I)$".
(51) Before Definition 21.1.1: "modules over a Dedekind quite nice" should be "modules over a Dedekind domain quite nice".
(52) 22.2.9: replace "$L = Rg$" with "$L^\vee = Rg$", and replace "$g^{-\otimes d/2}$" with "$g^{\otimes d/2}$" (to make the notation consistent).
(53) 22.3.2: should be "$x \otimes x \otimes g - g(Q(x))$"—no need to tensor with 1.
(54) Example 22.3.26: In first line should be "$R = \mathbb{Z}_F = \mathbb{Z}[\sqrt{10}]$", and at the bottom of the page should be "with $\operatorname{disc} B = (2+\sqrt{10})R$, so $\operatorname{Ram} B = \{(2,\sqrt{10}), (3, 2+\sqrt{10}), \infty_1, \infty_2\}$".
(55) Proof of Lemma 22.3.52: should be "we may suppose $f_1 = 0$" (not $e_1 = 0$).
(56) (22.1.4), and sentence below: should be "$\operatorname{nrd}^\sharp(\mathcal{O})$", and the right-hand side should be multiplied by $1/N$.
(57) 23.5.7: should be "$L' \subsetneq \pi L \subsetneq L'$".
(58) Proof of Proposition 23.5.8, at the end should be "By adjacency, $L_i \subsetneq \pi L_{i-1} \subseteq L_{i+1}$."
(59) Proposition 24.5.14(a): 'ramified' and 'split' are reversed.
(60) 26.2, "let d_F be the discriminant of F": "let d_F be the absolute discriminant of F (i.e., the absolute value of the discriminant of $\mathbb{Z}_F$)".
(61) (26.2.4), 26.2.14, (26.2.17), (26.2.20), (26.2.21), (26.8.4), (26.8.6), (26.8.13), (29.8.25), proofs of Lemma 29.8.24 and Corollary 29.10.3: remove absolute value bars so e.g. just "$\sqrt{d_F}$".
(62) (26.3.13): the final power should be "$\mathrm{N}(\mathfrak{p})^{2es}$" (not "$\mathrm{N}(\mathfrak{p})^{2s}$").
(63) 26.5, "(and $d_F > 0$)": delete.
(64) Theorem 26.5.4: the expression for the mass is missing a factor $\zeta_F(2)$.
(65) After (26.6.6), add "When $(\mathcal{O} \mid \mathfrak{p}) = *$, we define $\lambda(\mathcal{O}, \mathfrak{p}) = 1$."
(66) Change the text surrounding Lemma 27.1.13 to the following.

> We have a natural embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_v$ for all $v \in \operatorname{Pl}\mathbb{Q}$, and this extends to a diagonal embedding $\mathbb{Q} \hookrightarrow \underline{\mathbb{Q}}$.
>
> **Lemma 27.1.13.** *The diagonal embedding $\mathbb{Q} \hookrightarrow \underline{\mathbb{Q}}$ is an injective ring homomorphism and the image is closed and discrete as a subring of $\underline{\mathbb{Q}}$.*
>
> Thus the inclusion map $\mathbb{Q} \hookrightarrow \underline{\mathbb{Q}}$ is continuous giving $\mathbb{Q}$ the discrete topology (as would be the case for any map with discrete domain).

(67) Proof of Lemma 27.1.13: Delete first sentence.
(68) In Lemma 27.2.5, change "an injective continuous group homomorphism" to "injective group homomorphism (which is continuous, giving $\mathbb{Q}^\times$ the discrete topology)".
(69) Last sentence of §27.4: replace with "Via the projection map $\underline{F}^{(1)} \to \underline{F}_{\not S}^\times$, although $F^\times \leq \underline{F}_{\not S}^\times$ may no longer be closed, so the quotient $\underline{F}_{\not S}^\times / F^\times$ need not be Hausdorff, the quotient is still quasi-compact (every open cover has a finite subcover)."
(70) Theorem 28.5.5: "$\operatorname{nrd}(\mathcal{O}^\times)$" should be "$\operatorname{nrd}(\widehat{\mathcal{O}}^\times)$".
(71) Corollary 28.6.8: should be "S-indefinite".
(72) Example 28.9.10: Replace "when $(\operatorname{Cl}_{G(O)} R)^2$ is trivial" with "when $(\operatorname{Cl}_{G(O)} R)^2 = \operatorname{Cl}_{G(O)} R$, i.e., when $\# \operatorname{Cl}_{G(O)} R$ is odd".
(73) Proof of Proposition 29.6.10: replace proof with:

> First consider (a), and suppose F is a finite extension of $\mathbb{Q}_p$. We seek to satisfy (29.4.14); the equation holds up to a constant, $\tau = c\mu$ for some $c \in \mathbb{R}_{>0}$, so we may choose appropriate f and x and solve for c. We choose f as the characteristic

function of R and $x = 0$, so that $f(0) = 1$. Then

$$\begin{aligned} f^\vee(x) &= \int_F f(y)\psi(xy)\,\mathrm{d}\tau(y) = c\int_F f(y)\psi(xy)\,\mathrm{d}\mu(y) \\ &= c\int_R \psi(xy)\,\mathrm{d}\mu(y). \end{aligned} \tag{29.6.11}$$

By character theory, we get $f^\vee(x) = 0$ unless $\psi(xy) = 1$ for all $x \in R$; equivalently $\mathrm{Tr}_{F|\mathbb{Q}_p}(xy) \in \mathbb{Z}_p$ for all $y \in R$, i.e., $x \in R^\sharp$ where

$$R^\sharp = \operatorname{codiff}(R) = \{x \in F : \mathrm{Tr}_{F|\mathbb{Q}_p}(xR) \in \mathbb{Z}_p\}.$$

On the other hand, if $x \in R^\sharp$, then $\int_R \psi(xy)\,\mathrm{d}\mu(y) = \int_R \mathrm{d}\mu(y) = \mu(R) = 1$. Thus $f^\vee$ is c times the characteristic function of $R^\sharp$. Plugging now into (29.4.14), we have

$$1 = f(0) = \int_F f^\vee(y)\,\mathrm{d}\tau(y) = c\int_F f^\vee(y)\,\mathrm{d}\mu(y) = c^2\int_{R^\sharp}\mathrm{d}\mu(y) = c^2\mu(R^\sharp)$$

so $c = \mu(R^\sharp)^{-1/2}$.

Let x_i be a $\mathbb{Z}_p$-basis for R with $x_i^\sharp$ the dual basis, giving a $\mathbb{Z}_p$-basis for $R^\sharp$. By Lemma 15.6.17 we have

$$\operatorname{disc}(R) = [R^\sharp : R]_{\mathbb{Z}_p}$$

so since $\mu(R) = 1$ by additivity we have

$$\mu(R^\sharp) = |\operatorname{disc}(R)|^{-1}.$$

It follows then that

$$\tau = c\mu = \mu(R^\sharp)^{-1/2}\mu = |\operatorname{disc}(R)|^{1/2}\mu$$

is self-dual.

Part (b) is proven in exactly the same way, but now $\operatorname{codiff}(R) = R$.

(74) Example 29.8.5: "let $d_F \in \mathbb{Z}_{>0}$ be the absolute discriminant of F", and remove absolute value bars in displayed equation.

(75) Proposition 30.7.4: the proof of Theorem 30.4.7 does say how to handle the normalizer group, and in fact it can be quite complicated! This should be deleted, and to keep numbering consistent, the first equation in the example that follows should be numbered.

(76) 30.8.1, and proof of Proposition 30.8.5: the containment $S \subseteq K_q$ only holds for the norm one group. So replace 30.8.1 with:

> To a nontrivial cyclic subgroup of $\mathcal{O}^\times/R^\times$, we associate the quadratic field K it generates over F. For example, we may have $K \simeq F(\zeta_{2q})$ for q the order of the cyclic subgroup; but we may also have $\gamma \in \mathcal{O}^\times$ with $\gamma^2 = u \in R^\times$, with $K \simeq F(\sqrt{u})$. Conversely, to a quadratic field $K \supseteq F$ embedded in B, we obtain a (possibly trivial) cyclic subgroup $(K^\times \cap \mathcal{O}^\times)/R^\times$.

And replace "K_q" by "K" in the proof of Proposition 30.8.5. (The rest of the argument is unchanged.)

(77) 31.1.19: the equality between reduced norm and index is not true in general. The statement (Proposition 31.4.4) holds for $\mathfrak{a} = \operatorname{nrd}(J)$. One can work with the index with the following additional clause: "Without loss of generality, by weak approximation we may suppose that $\mathcal{O}'_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for all $\mathfrak{p}$ dividing the level $\mathfrak{M}$ of $\mathcal{O}$ and $\mathcal{O}'$."

(78) Example 31.8.5: "absolute discriminant d_F".

(79) 33.2.4: Replace "A **geodesic** is a continuous map $(-\infty, \infty) \to X$" with "A **geodesic** is the image of a continuous map $(-\infty, \infty) \to X$ such that the restriction to sufficiently small compact intervals defines a geodesic segment".
(80) Proof of Lemma 33.4.11: replace with "The proof is direct; it is requested in Exercise 33.5."
(81) Definition 34.8.4(ii): should be "$\phi_j = \psi_{ij} \circ \phi_i$".
(82) In 34.8.6, "f_{ij}" should be "ψ_{ij}".
(83) Lemma 36.2.8: the proof is not 'identical', since we would need $c \in \mathbb{R}^\times$. But here is how to reduce to that case, replacing the proof: "Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. We claim we may reduce to the case $c = 1$. Indeed, if $a = 0$, multiply on the left by an element of N to get $a \neq 0$; but then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & (1-b)/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -(1+c)/a & c \\ 1 & -a \end{pmatrix}.$$

Now repeat the first matrix calculation in Lemma 33.4.4, with $c = 1$."
(84) 36.5.1: angles sum to π, not 2π.
(85) 36.5.19: the series expansion for $\mathcal{L}(\theta)$ should have $(2\theta)^{2n}$ (not $2n+1$ in the exponent), and this equation should be numbered.
(86) 37.2.5: hyperbolic metric is missing a factor 2.
(87) 37.3.10: the first paragraph should conclude "next vertex $v_{m+1} = v_1$".
(88) 39.1.6: replace first sentence with "Let F be number field of degree $n = [F : \mathbb{Q}]$ with absolute discriminant d_F and r real places and c complex places, so $r + 2c = n$."
(89) Proof of Lemma 40.1.7: first line should be "This is true for $z \in$ ⌑ (where ⌑ is the standard fundamental domain described in 35.1.3), since then".
(90) (40.1.15): k should be $k-2$, so the equation should read

$$\wp(z) = \frac{1}{z^2} + \sum_{k=3}^{\infty} (k-1) G_k(\Lambda) z^{k-2} = \frac{1}{z^2} + 3G_4(\Lambda) z^2 + 5G_6(\Lambda) z^4 + \dots.$$

(91) Definition 40.4.3, and just before: replace two occurrences of "A" by "$[T]$".
(92) Example 41.1.5: replace "$T(p)_{11}$" by "$T(n)_{11}$".
(93) Example 41.1.11: replace "$\theta_{ij}(q)$" by "$\Theta_{ij}(q)$".
(94) Proof of Proposition 42.1.9: add ", a definite quaternion algebra by Lemma 42.1.5" to first line.
(95) Proof of Proposition 42.1.9: Before "Since $\mathcal{O}$ is a free $\mathbb{Z}$-module", add "Let $\ell \neq p$ be prime."
(96) Proof of Proposition 42.1.9: replace penultimate paragraph by:

> To conclude, we show that $\mathcal{O}_p$ is the valuation ring (13.3.3) of B_p and is therefore maximal (Proposition 13.3.4). Since $\mathcal{O}_{(p)}$ is dense in $\mathcal{O}_p$, it suffices to show that $\mathcal{O}_{(p)} = \{\alpha \in B : v(\alpha) \geq 0\}$. For $(\subseteq)$, if $\alpha \in \mathcal{O}_{(p)}$ then $\deg \alpha \in \mathbb{Z}_{(p)}$ so α is in the valuation ring. For $(\supseteq)$, let $\alpha \in B$ be a rational isogeny with $v(\alpha) \geq 0$, and write $\alpha = a\phi$ where ϕ is an (actual) isogeny not divisible by any integer. Then $v(\alpha) = \mathrm{ord}_p(a) + v(\phi) \geq 0$ and $0 \leq v(\phi) \leq 1/2$, since multiplication by p is purely inseparable; so $\mathrm{ord}_p(a) \geq -1/2$ and therefore $a \in \mathbb{Z}_{(p)}$, and hence $\alpha \in \mathcal{O}_{(p)}$.

(Cleaned up a bit, related localization to completion.)
(97) Proof of Lemma 42.2.13: the expression for $E[I\beta]$ holds when $\mathrm{nrd}(I\beta)$ is coprime to p; otherwise, this should be interpreted as a scheme-theoretic kernel.
(98) Proof of Proposition 42.2.16(b), "same right $\mathcal{O}'$-ideal class" should be "same left $\mathcal{O}'$-ideal class".

(99) Proof of Lemma 42.2.7: factoring through ϕ_I is not the definition of I! This statement follows from Proposition 42.2.16(b), so one could borrow from the future. Or see the addenda item below.
(100) (42.2.18): "$\operatorname{rk} E[I']$" should be "$\operatorname{rk} E_I[I']$".
(101) Proof of Theorem 42.3.2, "Tensoring with $\mathbb{Q}$... we may suppose $I_0 \subseteq B_0$": all occurrences of I_0 should be replaced by I.
(102) (42.3.5): "$(I' : I)$" should be "$(I : I')$".
(103) Before 42.3.10: delete ε_2 and ε_3 (their values are already given), so the formula reads

$$\frac{p-1}{12} + \frac{1}{4}\left(1 - \left(\frac{-4}{p}\right)\right) + \frac{1}{3}\left(1 - \left(\frac{-3}{p}\right)\right).$$

(104) Example 42.3.11: in second paragraph, should be "$\mathcal{O}_{\mathsf{L}}(I)^{\times} = \langle 1/2 - i(1+j)/4 \rangle$".
(105) below (43.2.18): Sentence should start "There exists a genus 2 curve over $\mathbb{Q}$".
(106) 43.5.7: I_6 is not holomorphic! So replace with "The functions I_4, I_{10} are holomorphic, but I_2, I_6 are meromorphic (poles as in Lemma 43.5.5)."
(107) 43.5.9: In the Albert classification, B is simple, so case (v) should not occur (and in case (iii), the quaternion algebra is a division algebra), so "five cases" should be "cases". See also the addenda below, which describes the split cases as well.

Exercises.

(1) Exercise 2.11: formatting on a) and b) is wrong, should match (a) and (b) in Exercise 2.9.
(2) Exercise 2.16: "βw" should be "$\operatorname{tr}(\lambda(\beta w))$".
(3) Exercise 3.6: "subfields" should be "quadratic subfields (over F)". (One does not need B to be a division quaternion algebra for the first statement.)
(4) Exercise 3.14: should be "$\operatorname{trd}(\alpha)$" not "$\operatorname{trd}(A)$".
(5) Exercise 3.18: replace "$V(B) =$" with "$V(B) :=$", and replace last two sentences "Let B be a ... over F" with "Let B be a *central* division ring over F. Show that $V(B)$ is a nonzero vector space if and only if B is a quaternion algebra over F."
(6) Exercise 4.5: V should be nondegenerate.
(7) Exercise 4.7: in (a), matrix should be transposed to get a left action; in (b), replace "$A[T]A^{\mathsf{t}}$" with "$A^{\mathsf{t}}[T]A$".
(8) Exercise 4.8(a): need $i' \neq 0$.
(9) Exercise 5.7: delete "$(-1, 26)_{\mathbb{Q}} = 1$, i.e.,", so the exercise is "Show $\left(\dfrac{-1, 26}{\mathbb{Q}}\right) \simeq \mathrm{M}_2(\mathbb{Q})$."
(10) Exercise 5.10(a): replace "$k \in \{i, j, ij\}$" with "$k \in B^0$". (Or keep this as is, then you can take $t = 0$ in the formulas after.)
(11) Exercise 5.12: replace "$^-\colon \operatorname{Clf}^0 Q \to \operatorname{Clf}^0 Q$" with "$^-\colon \operatorname{Clf} Q \to \operatorname{Clf} Q$".
(12) Exercise 5.15: move to end of chapter 12.
(13) Exercise 5.22: "R-algebra" should be "F-algebra".
(14) Exericse 5.23: "an linear" should be "a linear".
(15) Exercise 6.12: "$\zeta^2 = 1$" should be "$\zeta^2 = d$".
(16) Exercise 7.6: "simple F-algebra".
(17) Exercise 7.8: "$(K \mid b)$" should be "$(K, b \mid F)$".
(18) Exercise 7.10: "show" should be "show directly".
(19) Exercise 7.15(c): The summation should be over $g \in G$, "g^{-1}" should be "$(g^{-1})^{\mathrm{o}}$", and "Give B the structure of a B^{e}-algebra" should be "Give B the structure of a B^{e}-module".
(20) Exercise 7.16: only one direction is true. So replace with "Let B be an F-algebra, and let F^{al} be an algebraic closure of F. Show that if $B \otimes_F F^{\mathrm{al}}$ is simple then B is simple, but give a counterexample to the converse."

(21) Exercise 7.20: add "(viz. Main Theorem 4.4.1)" at end.
(22) Exercise 7.23: "Exercise 7.18" should be "Exercise 7.17" and add "central" to the first sentence.
(23) Exercise 7.24: "let $f(T) \in K[T]$" should be "let $f(T) \in K[T]$ be monic".
(24) Exercise 9.8: replace $\mathfrak{p}$ with $\mathfrak{m}$.
(25) Exercise 10.8: "$R = S[\alpha]$ a" should be "$R = S[\alpha]$ is a".
(26) Between Exercises 11.3 and 11.4: the one starting "Check that the map" should be a separate exercise; so the numbering of all of the remaining ones should increase by one.
(27) Exercise 11.8 (appears as 11.7): should be "such that $\|x - \lambda\|^2 \leq 1/2$".
(28) Exercise 11.9 (appears as 11.8): delete "definite", and replace "$\mathcal{O} \subseteq B$" with "$\mathcal{O} \subset B$".
(29) Exercise 11.10(c) (appears as 11.9(c)): replace exercise with: "More generally, if F is a field of characteristic 2 show that there is an exact sequence

$$1 \to F^2 \to \mathrm{Aut}_F(\mathcal{O} \otimes_{\mathbb{Z}} F) \to K^\times \rtimes \mathrm{Aut}_F(K) \to 1$$

where $K := F[\omega] \simeq F[x]/(x^2 + x + 1)$, and F^2 is considered as an additive group. *[Hint: let $J = \mathrm{rad}(\mathcal{O} \otimes_{\mathbb{Z}} F)$ be the Jacobson radical of the algebra, and show that the sequence is induced by an F-linear automorphisms of $K := F[\omega]$ and the automorphisms $\omega \mapsto \omega + \epsilon$ with $\epsilon \in J$.]*"
(30) Exercise 11.13 (appears as 11.12): replace "is conjugate in O(2) to" to "is".
(31) Exercise 11.16(b) (appears as 11.15(b)): replace "$x^2 + y^2 + z^2 = p$ with $x, y, z \in \mathbb{Z}$" with "$t^2 + x^2 + y^2 + z^2 = p$ with $t, x, y, z \in \mathbb{Z}$".
(32) Exercise 25.5: should say $w := \#\mathcal{O}^\times/\{\pm 1\}$ (missing $\times$).
(33) Exercise 33.7: in the proof of Theorem 33.5.5, refer to Exercise 33.7(d), and replace the exercise as follows:
"In this exercise, we consider the action of $\mathrm{PSL}_2(\mathbb{R})$ on points and geodesics in $\mathbf{H}^2$.
  (a) Show that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on the set of geodesics in $\mathbf{H}^2$.
  (b) Show that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on the set of geodesics in $\mathbf{H}^2$ of a fixed length. *[Hint: using (a), reduce to the case where all four endpoints lie on the imaginary axis. Use elements of A in* (33.4.1) *to map one endpoint each to i; then use an element of K.]*
  (c) Show that every orientation-preserving isometry of $\mathbf{H}^2$ that maps a geodesic to itself and fixes two points on this geodesic is the identity.
  (d) Conclude that for every isometry ϕ of $\mathbf{H}^2$ and every geodesic in $\mathbf{H}^2$, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $g\phi$ fixes the geodesic pointwise."
(34) Exercise 36.12: add new part (a), "Prove that $|B_{2k}| = (-1)^{k+1} B_{2k}$ for $k \geq 1$.

Typos/copyediting.

(1) Section 1.1, line 1: add attribution for "most famous act of mathematical vandalism" to Gordon–McNulty.
(2) Section 2.1, line 5: replace with "$G^n := \{g^n : g \in G\} \leq G$ for the subgroup of nth powers".
(3) Section 2.1, line 7: space missing in "B equipped".
(4) Section 2.1, page 22, line 2: replace "notation. reserve" with "notation. We reserve".
(5) Section 2.1, page 22, line 4: replace "$\mathrm{End}_F(B) \sim \mathrm{M}_n(F)$" with "$\mathrm{End}_F(B) \simeq \mathrm{M}_n(F)$".
(6) 3.2.9, line 7: delete extraneous "$\overline{ij}$".
(7) Section 4.1, line 10: "element respect to" should be "element with respect to".
(8) Proof of Main Theorem 4.4.4: end with "The converse follows from Example 4.3.8".
(9) Proof of Corollary 4.4.5: replace with "The first statement is immediate. The second follows as in the proof of Main Theorem 4.4.1: we may suppose B is a quaternion algebra

and $K = F[i]$, and we proved that the centralizer of K in B is K, so the centralizer of $K^\times$ in $B^\times$ is $K^\times$."

(10) Before 4.5.8, replace "Since a reflection ... *even number of reflections*" with "Since reflections have determinant -1, we have $f \in \mathrm{SO}(Q)(F)$ if and only if f is the product of an *even* number of reflections."
(11) Exercise 4.4: a) and b) should be (a) and (b).
(12) Exercise 4.8: "readers some" should be "readers with some".
(13) Exercise 4.16(a): in hint, "complement of V" should be "complement in V".
(14) (5.3.3): line after, "two-sided ideal generated the" should be "two-sided ideal generated by the".
(15) Definition 4.2.12: delete preceding sentence "From now on... associated to $\mathbb{Q}$".
(16) Remark 4.2.19: "as least as old" could be "at least as old".
(17) Exercise 5.2: second "(a)" should be "(b)".
(18) Section 5.1, line 6, "But always have scalar norms" should be "But we always have scalar norms".
(19) Before Lemma 6.3.7, "Next, even though not every quadratic form" should be "Next, not every quadratic form".
(20) Section 7.7: Replace "We conclude this chapter with" with "In this section, we establish".
(21) Corollary 7.7.6: in the line before, replace "*isomorphism classes* of quaternion algebras" with "*isomorphism classes* of quaternion algebras (also proven in Exercise 6.4, in a different way)."
(22) 7.5.5, 7.7.12: Exercises are numbered wrong, "Exercise 7.12" should be "Exercise 7.11", "Exercise 7.11" should be "Exercise 7.10", and probably others.
(23) Section 7.8: "In this last section, we" should be "We now", and delete "Thr".
(24) Proof of Lemma 7.8.5: delete "For part (a)".
(25) Lemma 7.8.8: "(as in the proof of Lemma 7.8.5" needs right parenthesis.
(26) Exercise 7.20(b): "(viz. Main Theorem mthm:nonsing)" should be "(viz. Main Theorem 4.4.1)".
(27) Before 8.2.7: "Laghbribi" should be "Laghribi".
(28) Remark 8.2.9: "[Lam2005, Example VI.1]" should be "[Lam2005, Example VI.1.11]".
(29) Before Remark 9.4.10: replace "Theorem 9.4.9 $M_{(\mathfrak{p})} = M'_{(\mathfrak{p})}$" by "Theorem 9.4.9, we have $M_{(\mathfrak{p})} = M'_{(\mathfrak{p})}$".
(30) Proof of Lemma 10.4.3: should be "$\bigcap_{\mathfrak{q}\neq\mathfrak{p}} \mathcal{O}_{(\mathfrak{q})}$".
(31) Proof of Lemma 10.4.4: should be "$\mathcal{O}'_{(\mathfrak{p})} = \mathcal{O}_{(\mathfrak{p})}$".
(32) Proof of Lemma 11.1.2: Replace "Then $\mathrm{trd}(\alpha) = 2t \in \mathbb{Z}$, so by Corollary **??** we have $t \in \frac{1}{2}\mathbb{Z}$" by "Then $\mathrm{trd}(\alpha) = 2t \in \mathbb{Z}$ by Corollary 11.1.3, so $t \in \frac{1}{2}\mathbb{Z}$".
(33) Below Figure 11.2.7: replace "four inscribed tetrahedra" by "four inscribed regular tetrahedra".
(34) 11.3.1: replace "left (or right)" with "(left or) right".
(35) Before Proposition 11.3.4: replace "for division on the left" with "on the left".
(36) Before Corollary 11.3.6: replace "exists a greatest common divisor" with "exists a right greatest common divisor".
(37) Proof of Proposition 11.3.4: replace "left Euclidean" with "right Euclidean".
(38) Corollary 11.3.6: replace "there exists" with "there exist".
(39) (11.4.9): "f or" should be "for".
(40) Below (11.4.10): should be "$\gamma_1, \dots, \gamma_{r-1} \in \mathcal{O}^\times$".
(41) Before Theorem 13.3.11: add "We recall the notation 6.1.5."
(42) p. 258, "both of these products are compatible": delete "are".

(43) References to Exercise 13.7 (23.2.5, Exercise 23.8, 42.4.6) should be to Exercise 13.8.
(44) Example 14.2.13: "qudaratic" should be "quadratic", and "not ramified at p" should be "not ramified at q".
(45) Example 15.5.7: "qudaratic" should be "quadratic".
(46) Remark 15.6.18, "instead of the codifferent instead a *different*": delete second "instead", add a comma.
(47) Before 16.2.5, replace "$B = FJ = F(rJ) \subseteq FIJ$" with "$B = F(rJ) \subseteq F(IJ) = B$"; and in the line before, replace "$r \in I$" with "$r \in R \cap I$".
(48) Proof of Proposition 16.4.4: replace sentence in second paragraph with "Then for all $\alpha \in I$, we have

$$[\mathcal{O} : I]_R [I : \mathcal{O}\alpha]_R = [\mathcal{O} : \mathcal{O}\alpha]_R = \mathrm{Nm}_{B|F}(\alpha) R,$$

the first equality holding by Lemma 9.6.4 (the index is given by the determinant of a change of basis)".
(49) p. 266, line -2, "Finally, not every lattice": delete "Finally".
(50) p. 280, line -2, "Lemma 17.3.3(b)": should be "Lemma 17.3.3(ii)".
(51) Proof of Corollary 17.2.3: "$R_{\mathfrak{p}}$ is a complete DVR".
(52) 17.3.7: specify that $B = \mathrm{M}_n(F)$.
(53) 17.4.15, "is given": should be "are given".
(54) Main Theorem 17.7.1: last sentence should end "are finite".
(55) p. 312, middle paragraph: missing parenthesis at end.
(56) Before Remark 19.5.8: "Brant" should be "Brandt".
(57) Before Theorem 20.1.1: delete extra space before "**projective**".
(58) End of Example 20.1.2: "invertible as lattice" should be "invertible as a lattice".
(59) Proof of Theorem 20.3.3: The two occurrences of "α_i" indicating a set should be "$\{\alpha_i\}_i$".
(60) Remark 20.3.6, line 2: delete extra space before "**dual basis lemma**".
(61) 22.3.1 "nondegenerateternary" should be "nondegenerate ternary".
(62) Example 22.3.26: in line 6, should be $\mathcal{O} = \mathrm{Clf}^0(Q) = R \oplus \mathfrak{p}^{-1}i \oplus \mathfrak{p}^{-1}j \oplus \mathfrak{p}^{-1}k$ (so all $\oplus$).
(63) 23.2.2: "have the nice local description" should be "have the following nice local description".
(64) Before (24.1.2), "proper implications": just "implications".
(65) Before (24.1.3), "$\mathcal{O}^\natural = \mathcal{O}_{\mathsf{L}}(\operatorname{rad}\mathcal{O})$": should be :=.
(66) 24.2.19, "qudratic" should be "quadratic".
(67) 24.3.1: "posibilities" should be "possibilities".
(68) Definition 24.3.2: space missing before "residually" in two places, and the "we say" line is not formatted in line with the rest.
(69) Proof of Proposition 24.5.14: Delete duplicate reference to 24.5.12.
(70) Before (25.3.14): Replace "Proposition 25.3.13" with "(25.3.7)".
(71) Before Lemma 26.4.1: Replace "every nonzero ideal is invertible" with "every nonzero $\mathcal{O}_{\mathfrak{p}}$-ideal is invertible (Proposition 16.6.15(b))".
(72) Proof of Lemma 26.6.7: space missing in "that if($\mathcal{O}'$".
(73) (27.6.13) and below: should be colonequals, and "$\underline{B}^1 := \{\underline{\alpha} \in \underline{B} : \mathrm{nrd}(\underline{\alpha}) = 1\}$."
(74) Before Theorem 28.2.11: replace "()" with "(Lemma 17.7.13)".
(75) Proof of Corollary 28.3.6: in the appeal to Lemma 28.7.2, add "borrowing (in a self-contained way) from the future".
(76) Above Corollary 28.5.10: replace "$\mathrm{Cl}_{G(\mathcal{O})}$" with "$\mathrm{Cl}_{G(\mathcal{O})} R$".
(77) Proof of Proposition 28.7.3: the reference to Exercise 7.31 should be Exercise 7.30.
(78) 29.6.7(b)–(c), proof of Proposition 29.6.10, 29.8.9: replace $F/$ with $F|$ in subscripts.
(79) Definition 29.6.8: write "$\psi = \psi_F$" (we immediately abbreviate).
(80) (30.8.4) and in the proof of Proposition 30.8.5: replace "$\mathrm{Emb}(S; \mathcal{O})$" with "$\mathrm{Emb}(S, \mathcal{O})$".

(81) Lemma 34.4.5: "primages" should be "preimages".
(82) Proof of Proposition 36.6.2, "and $y' = y/\|z\|^2 > y$, and we repeat": replace with "and $y' = y/\|z\|^2 > y$, and so $\|\gamma z\|^2 \geq 1$".
(83) 37.2.9: in the subscript of $\bigcap$, replace $\Gamma-$ with $\Gamma\setminus$.
(84) Before Definition 39.4.10, bottom of page 738: delete extraneous $\square$ in $[\mathrm{Cl}_\Omega R : \mathrm{Cl}\, R]$.
(85) After (41.5.6): delete indent in front of "where $\delta = 1, 0$".
(86) Proof of Lemma 42.1.11: should be "[Sil2009, Exercise V.5.4(b)]" (instead of "[Sil2009, Exercise III.5.4(b)]").
(87) After (42.2.2): "$E[\alpha] = \ker\alpha$" should be "$E[\alpha] := \ker\alpha$".
(88) Proof of Lemma 42.2.22: at the end of the proof, "..'' should be ".".
(89) (42.2.27): "$\mathrm{Hom}(E_{I'} E_I)$" should be "$\mathrm{Hom}(E_{I'}, E_I)$".
(90) Symbol Definition List: "Eichler symbol of a local" should conclude with "order".
(91) Bibliography: the items [Hur1896] and [Hur1898] should be interchanged.

Addenda

(1) Remark 8.2.9: Albert's book [Alb39] on algebras still reads well today. The proof of the key implication (iii) $\Rightarrow$ (i) in Proposition 8.2.3 is due to him [Alb72]. ("I discovered this theorem some time ago. There appears to be some continuing interest in it, and I am therefore publishing it now.") Albert [Alb32] used Proposition 8.2.8 to show that over $F = \mathbb{R}(x, y)$, the tensor product of

$$B_1 = \left(\frac{x, -1}{F}\right) \quad \text{and} \quad B_2 = \left(\frac{-x, y}{F}\right)$$

is a division algebra, by verifying that the Albert form $Q(B_1, B_2)$ is anisotropic over F. See Lam [Lam2005, Albert's Theorem 4.8, Example VI.1.11] for more details.

For the fields of interest in this book (local fields and global fields), a biquaternion algebra will never be a division algebra—the proof of this fact rests on classification results for quaternion algebras over these fields, which we will take up in earnest in Part II.
(2) Proof of Lemma 11.4.1: replace second sentence with: "Then $\mathcal{O}/p\mathcal{O} \simeq (-1, -1 \mid \mathbb{F}_p) \simeq \mathrm{M}_2(\mathbb{F}_p)$ by Wedderburn's little theorem. There exists a right ideal $I \bmod p \subset \mathcal{O}/p\mathcal{O}$ with $\dim_{\mathbb{F}_p}(I \bmod p) = 2$, for example $I \bmod p = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$."
(3) Definition 24.3.2: add "We say that $\mathcal{O}$ has **Eichler invariant** given by the Eichler symbol $\left(\frac{\mathcal{O}}{\mathfrak{p}}\right)$".
(4) Example 28.5.20: Let F be a number field and let B be an indefinite quaternion algebra over F (so either F has a complex place or at least one real place of F is unramified in B). Suppose that $R = \mathbb{Z}_F$ has narrow class number 1, and let $\mathcal{O} \subseteq B$ be an Eichler R-order in B. Then $\#\,\mathrm{Cls}\,\mathcal{O} = 1$. Indeed, we apply Theorem 28.5.5: by Example 28.5.16, the order $\mathcal{O}$ is locally norm-maximal so $\mathrm{Cl}_{G(\mathcal{O})} R$ is a quotient of the narrow class group, which is trivial.
(5) Remark 33.2.8: at the end of the first paragraph, add "For an approach geared towards the context of hyperbolic geometry, see Ratcliffe [R."
(6) Before Definition 34.4.7: should refer to Lemma 34.4.1(iii) (not (iv)).
(7) Remark 37.2.11: clarify first sentence "In the identification $\mathbf{H}^2 \to \mathbf{D}^2$, the preimage of an isometric circle in $\mathbf{D}^2$ is the corresponding perpendicular bisector, since this identification preserves hyperbolic distance."
(8) 37.3.3: replace first paragraph with: "We recall Definition 33.6.5 (sides and vertices) for hyperbolic polygons. For a Dirichlet domain $\mathbin{\text{⌑}}$, a side is a geodesic segment of positive

length of the form $¤ \cap \gamma ¤$ with $\gamma \in \Gamma \smallsetminus \{1\}$; and a vertex is the point of intersection between two sides, equivalently, a vertex is a single point of the form $¤ \cap \gamma ¤ \cap \gamma' ¤$ with $\gamma, \gamma' \in \Gamma$."

(9) Lemma 42.2.7 can be proven by appeal to the Isogeny Theorem, as follows.

The image of $\mathrm{Hom}(E_I, E)$ under precomposition by ϕ_I lands in $\mathrm{End}(E) = \mathcal{O}$. We check locally that the image is I. First, we may replace I by an ideal in the same left $\mathcal{O}$-ideal class to suppose that $\mathrm{nrd}(I)$ is coprime to p. Then $I_p = \mathcal{O}_p$. For the remaining primes, let $\ell \neq p$ be prime. As in the proof of Lemma 42.1.11, the Isogeny Theorem gives

$$\mathrm{Hom}(E_I, E) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}(T_\ell(E_I), T_\ell(E))$$

(recalling that over a sufficiently large finite subfield of F, the Galois action is scalar). Since I is locally principal, we have $I_\ell = \mathcal{O}_\ell \alpha_\ell$ for some $\alpha_\ell \in \mathcal{O}_\ell \simeq \mathrm{M}_2(\mathbb{Z}_\ell)$ with $T_\ell(E) = \mathbb{Z}_\ell^2$. Then $T_\ell(E_I) = \alpha_\ell^{-1} T_\ell(E)$ and so

$$\mathrm{Hom}(T_\ell(E_I), T_\ell(E)) = \mathcal{O}_\ell \alpha_\ell.$$

The pullback map

$$\mathrm{Hom}(T_\ell(E_I), T_\ell(E)) \to \mathcal{O}_\ell$$

is just the identity map, since we are already writing isogenies with respect to the fixed (standard) basis of $T_\ell(E)$; so its image is $\mathcal{O}_\ell \alpha_\ell = I_\ell$. Therefore the image lies in I by the local–global dictionary for lattices.

(10) Delete sentence after Lemma 42.1.11, and add the following as Corollary 42.2.28 at the end of section 42.2.

Corollary 42.2.28. For all E' supersingular, there exists a separable isogeny $\phi\colon E \to E'$.

Proof. By Corollary 42.2.21, we have $E' \simeq E_I$ for a left $\mathcal{O}$-ideal I, which by Lemma 42.2.7 is well-defined on the left ideal class of I. The result follows then by Exercise 17.5: we may choose a representative $I' \sim I$ with $\mathrm{nrd}(I')$ coprime to p, so there is $\beta \in I'$ with $\mathrm{nrd}(\beta)$ coprime to p, yielding the desired separable isogeny.

In fact, more is true: by Proposition 28.5.18 (spelled out in Example 28.5.19), we may choose the separable isogeny to have degree supported in any nonempty set of primes not containing p. ☐

(11) Exercise 42.5: In the proof of Proposition 42.2.16, we considered $II' = \mathcal{O}\alpha$ and the isogeny $\phi_{I'}\colon E_I \to E_I/E_I[I']$, which moves away from the setup with the fixed supersingular elliptic curve E. We may proceed differently as follows.
   (a) Let $m := \mathrm{nrd}(I)$. From $I\overline{I} = \mathcal{O}m$ show that $\phi_{\overline{I}} = \phi_I^\vee$ (dual isogeny). Conclude that $\deg \phi_I = \deg \phi_{\overline{I}}$.
   (b) Prove $\deg \phi_{I'} \mid \mathrm{nrd}(I')$ by working with $\phi_{\overline{I'}}\colon E \to E_{\overline{I'}}$.

(12) 43.5.9: Replace with the following.

Let A be a principally polarized complex abelian surface. Let $\mathrm{End}(A)$ be the ring of endomorphisms of A, and let $B = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $A \sim E_1 \times E_2$ is isogenous to the product of two elliptic curves, then either $E_1 \not\sim E_2$ are not isogenous and $B \simeq \mathrm{End}(E_1) \times \mathrm{End}(E_2)$ or $E_1 \sim E_2 \sim E$ and $B \simeq \mathrm{End}(E^2) \simeq \mathrm{M}_2(\mathrm{End}(E))$. As the endomorphism algebra of an elliptic curve is either $\mathbb{Q}$ or an imaginary quadratic field K, this gives four possibilities: $B \simeq \mathbb{Q}\times\mathbb{Q}, \mathbb{Q}\times K, \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(K)$. Otherwise, B is simple, and by the classification theorem of Albert (Theorem 8.5.4), the $\mathbb{Q}$-algebra B is exactly one of the following:
   (a) $B = \mathbb{Q}$, and we say A is **typical**;
   (b) $B = F$ a real quadratic field, and we say A has **real multiplication (RM)** by F;
   (c) B is an indefinite division quaternion algebra over $\mathbb{Q}$, and we say A has **quaternionic multiplication (QM)** by B; or

(d) $B = K$ is a quartic CM field K, and we say A has **complex multiplication (CM)** by K.

One may also view the products $B \simeq \mathbb{Q} \times \mathbb{Q}$ and $B \simeq \mathrm{M}_2(\mathbb{Q})$ as special cases of (ii) and (iii), respectively.

References

[1] John Voight, *Quaternion algebras*, Grad. Texts in Math., vol. 288, Springer, Cham, 2021.